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SOME EFFICIENT COMPUTATION TECHNIQUES INCLUDING THEIR APPLICATION TO TIME OPTIMAL TRAJECTORIES FROM PARKING ORBIT

by William B. Tucker

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DEFINITION OF SYMBOLS

Symbol	Definition
$\overline{R}$	Position vector of the vehicle
R	Magnitude of the position vector of the vehicle
$\hat{R}$	Unit vector in the direction of the vehicle's position vector, i.e. $\overline{R} = R \cdot \hat{R}$.
$\overline{V}$	Velocity vector of the vehicle, or the time rate of change of $\overline{R}$.
$\overline{A}$	Acceleration vector of the vehicle; the time rate of change of $\overline{V}$.
V	Magnitude of the velocity vector
$\overline{\lambda}$	Control variable vector defining the thrust direction
$\dot{\overline{\lambda}}$	Time rate of change of $\overline{\lambda}$, the control variable vector
Λ	Magnitude of $\overline{\lambda}$, i.e. $\Lambda = \lvert\overline{\lambda}\rvert$
$\dot{\Lambda}$	Time rate of change of $\lvert\overline{\lambda}\rvert$
$\hat{\lambda}$	Unit vector defining the direction of the thrust
$\dot{\hat{\lambda}}$	Time rate of change of $\hat{\lambda}$
$\ddot{\overline{\lambda}}$	The control variable differential equations; time rate of change of $\dot{\overline{\lambda}}$
$\overline{R} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \overline{V} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \overline{A} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$	Position, velocity, and acceleration components along any space-fixed coordinate axes, x, y, and z.
X_i	Indicates X for i = 1, Y for i = 2, and Z for i = 3.
$\hat{\omega}$	Unit vector along the rotation vector of the earth i.e. along the north polar axis of the earth
ϕ'	Latitude angle of the vehicle, i.e. $\sin \phi' = \hat{R} \cdot \hat{\omega}$

Symbol	Definition
J, H, D	Oblateness coefficients of the accepted earth model
GM	Gaussian gravitational constant
A	Equatorial radius of the earth
U	Potential function
∇	The DEL operator; for example $-\nabla U = \ddot{\overline{R}}g = -\frac{\partial U}{\partial X_i}$, i = 1, 2, 3.
$\overline{\beta}$, $[\overline{\beta}]$	Denote a column vector and row vector, respectively, i.e. $\overline{\beta} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $[\overline{\beta}] = [b_1, b_2, b_3]$.
$\overline{R} \times \overline{V}$	Vector cross-product of $\overline{R}$ onto $\overline{V}$, i.e. angular momentum vector.
$\overline{R} \cdot \overline{V}$	Vector scalar-product of $\overline{R}$ and $\overline{V}$, i.e. $\overline{R} \cdot \overline{V} = R\, V \cos(\overline{R}, \overline{V})$.
α_n	Angle of attack measured from the velocity vector to the projection of the thrust direction in the flight plane, positive in the direction of motion (positive nose-down).
α_w	Out of plane angle of attack, measured from the projection of the thrust vector in the flight plane to the thrust vector, positive toward the angular momentum vector.
$\dot{\alpha}_n$, $\dot{\alpha}_w$	Time rate of change of α_n and α_w.
$\overline{S}$	Vector in the direction of the perifocus and of magnitude equal to the eccentricity of the instantaneous conic section.
e	Eccentricity of a conic section.
$\hat{S}$, $\hat{T}$, $\hat{R}$	An orthogonal right-hand system of unit vectors, $\hat{S}$ being parallel to the incoming asymptote of an hyperbola, $\hat{T}$ being in the plane normal to the $\hat{S}$ in some convenient direction (for example in the moon's travel plane), and $\hat{R}$ being defined by the following vector cross product: $\hat{R} = \hat{S} \times \hat{T}$.

SOME EFFICIENT COMPUTATION TECHNIQUES INCLUDING THEIR APPLICATION TO TIME OPTIMAL TRAJECTORIES FROM PARKING ORBIT

SUMMARY

The amount of time and effort expended in generating time optimal trajectories is presently large and increasing daily. This emphasizes the need for efficient techniques for computing such trajectories.

Presented in this report is the development of some efficient techniques for generating time optimal trajectories from parking orbit to desired terminal conditions. Each technique is illustrated with a specific detailed application to the lunar-interplanetary trajectory problem (assuming one powered phase from parking orbit to injection into the transit conic).

The techniques developed are in the following general areas of interest: formulation of the differential equations for the control variables; parametric representation of the control variables, transformations relating the various parameters, and the efficient use of these parameters; transversality conditions, with emphasis on their application to the lunar-interplanetary cutoff surface constraints; transformation of mission objectives into target arrival conditions that are efficient for use in multivariable isolation schemes.

Fundamental, motivating discussions are included to aid in adapting the techniques or principles to the solution of other problems.

SECTION I. INTRODUCTION

Presented in this report are some efficient techniques for computing time optimal trajectories from parking orbit to a target. The target is treated first as a surface or hypersurface defined by functional terminal contraints. Later, the targeting problem is treated where the desired terminal constraints are defined by mission objectives.

It is not the object of this report to present a theoretical calculus of variations treatment of the problem at hand. Rather, the objective is to interpret the results of such a treatment and to develop in detail some efficient techniques for applying the theory in trajectory computation and analysis. This report uses the theoretical results as basis to develop efficient techniques.

References 1, 2, 6, and 7 present the generalized theoretical treatment of variational problems and Reference 3 presents the development for the particular problem to be considered in this report.

It is hoped that the ideas used to develop the techniques presented here will be adapted to other problems to save manhours and computer time. But, aside from any other application that might be made, the application that is given in detail here - that of generating time optimal trajectories from a parking orbit to various defined targets - has already rendered large savings at MSFC.

SECTION II. FUNDAMENTAL INTERPRETATION OF THE PROBLEM

The basic problem to be treated in this report is the transfer of a vehicle from a given point in three dimensional Euclidean space to some variable terminal point in that space. The points in space are defined by so-called state variables. Thus, the initial point is specified by six quantities defining position and velocity. The terminal point is variable, but defined functionally in terms of vehicle position and velocity components. The terminal functions must have certain characteristics for the theory to be applicable [Ref. 1], but those treated in this report have the proper character.

For this report the vehicle is assumed to have constant thrust, and constant mass flow rate. All atmospheric forces are neglected. The earth's potential is assumed to contain the spherical term and the second, third and fourth harmonics.

A calculus of variations treatment of the subject problem (outlined previously) results in six so-called control variables. From the homogeneity of the differential equations, it can be shown that there are really only five independent control variables. For this problem the control variables can be physically interpreted as the thrust vector direction, and the rate of change of the thrust vector direction. The calculus of variations treatment of the problem results in differential equations that uniquely define the acceleration of the thrust vector direction in terms of the initial conditions of the problem.

For this report the problem is resolved to the determination of initial conditions for the control variables such that the time required to transfer the vehicle from the specified set of initial state variables to some desired set of terminal state variables is a minimum. This may sound rather simple but it presently requires a large amount of computer time.

SECTION III. EFFICIENT FORMULATION OF THE DIFFERENTIAL EQUATIONS

The efficiency gained by the formulation of the differential equations as presented here comes about in three ways:

1. The control variable differential equations, i.e. $\ddot{\lambda}$, are separated into a "dominant" part, $\ddot{\lambda}_D$, and a "minor" part, $\ddot{\lambda}_M$. The terms "dominant" and "minor" are used because the customary terms such as first-order, perturbing, or second-order may not be strictly applicable.

Normally the separation is made such that the spherical earth part is separated from the part due to oblateness terms, but it is shown that since the gravitation acceleration contains the oblateness terms, much more of the oblateness contribution to $\ddot{\lambda}$ can be gotten at essentially no cost.

2. One of the reference vectors, $\hat{\omega}$ (rotational vector of the earth), is invariant with respect to time.

3. The formulation is applicable or usable in any spacefixed coordinate system. This is perhaps more properly termed flexibility than efficiency.

To obtain full benefit of the formulation, the acceleration of the vehicle due to gravity, $\bar{A}_g$ or $\ddot{\bar{R}}_g$, must be properly formulated, as is shown in the following development.

A. ACCELERATION DUE TO GRAVITY

The potential function associated with the accepted earth model contains the second, third, and fourth harmonies. It can be found in various texts in various forms. The form used here is

$$mU = \frac{-mGM}{R}\left[1 + \frac{JA^2}{3R^2}\left(1 - 3\sin^2\phi'\right) + \frac{H\,A^3}{5\,R^3}\left(3 - 5\sin^2\phi'\right)\sin\phi' + \frac{D\,A^4}{35\,R^4}\left(3 - 30\sin^2\phi' + 35\sin^4\phi'\right)\right]$$

Defining $U = -\frac{GM}{R}\,[1 + \sigma]$, then

$$\ddot{X}_{gi} = -\nabla U = \left[\frac{\partial}{\partial X_i}\left(\frac{GM}{R}\right)(1 + \sigma) + \frac{GM}{R}\frac{\partial}{\partial X_i}(\sigma)\right] e_i \text{ where } e_i \text{ indicates}$$

a unit vector in the positive X_i direction.

The terms to be differentiated are of two basic forms, namely $\frac{K}{R^n}$ and $c \sin^n \phi'$. Applying the chain rule $\left(\frac{\partial f}{\partial X_i} = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial X_i}\right)$ one finds that

$$\frac{\partial}{\partial X_i}\left(\frac{K}{R^n}\right) = -n\left(\frac{K}{R^{n+1}}\right)\frac{X_i}{R} = -\frac{n}{R}\frac{K}{R^n}\cdot\frac{X_i}{R},$$

and, since $\sin\phi' = \hat{R}\cdot\hat{\omega}$,

$$c\,\frac{\partial}{\partial X_i}\left(\sin^n\phi'\right) = \frac{nc}{R}\left(\sin^{n-1}\phi'\,\omega_i - \sin^n\phi'\,\frac{X_i}{R}\right).$$

Using these partials, then, and rearranging terms, one can write

$$\ddot{R}_g = \frac{GM}{R^2}\Bigg[\Bigg\{-1 - J\left(\frac{A}{R}\right)^2\left[(1-3\sin^2\phi') - 2\sin^2\phi'\right]$$

$$-\frac{H}{5}\left(\frac{A}{R}\right)^3\left[4\sin\phi'(3-5\sin^2\phi') + 3\sin\phi' - 15\sin^3\phi'\right]$$

$$-\frac{D}{7}\left(\frac{A}{R}\right)^4\left[(3-30\sin^2\phi'+35\sin^4\phi') + (4)7\sin^4\phi' - 12\sin^2\phi'\right]\Bigg\}\hat{R}$$

$$+\Bigg\{-\frac{6J}{3}\left(\frac{A}{R}\right)^2\sin\phi' - \frac{H}{5}\left(\frac{A}{R}\right)^3(15\sin^2\phi' - 3) - \frac{D}{35}\left(\frac{A}{R}\right)^4(60\sin\phi' -$$

$$-(4)(35)\sin^3\phi')\Bigg\}\hat{\omega}\Bigg],$$

or

$$\ddot{R}_g = -\frac{GM}{R^2}\Bigg\{\Bigg[1 + J\left(\frac{A}{R}\right)^2(1-5\sin^2\phi') + H\left(\frac{A}{R}\right)^3(3-7\sin^2\phi')\sin\phi'$$

$$+3D\left(\frac{A}{R}\right)^4\left(\frac{1}{7} - 2\sin^2\phi' + 3\sin^4\phi'\right)\Bigg]\hat{R}$$

$$+\Bigg[2J\left(\frac{A}{R}\right)^2\sin\phi' + 3H\left(\frac{A}{R}\right)^3\left(\sin^2\phi' - \frac{1}{5}\right) + 4D\left(\frac{A}{R}\right)^4\sin\phi'\left(\frac{3}{7} - \sin^2\phi'\right)\Bigg]\hat{\omega}\Bigg\}.$$

Defining the following terms,

$$\hat{R} = \frac{\overline{X}}{R} = \frac{X_i}{R}, \quad \hat{\omega} = \omega_i, \quad (i = 1, 2, 3)$$

$$S_J = (1 - 5 \sin^2 \phi')$$

$$S_H = (3 - 7 \sin^2 \phi')$$

$$S_D = (\frac{1}{7} - 2 \sin^2 \phi' + 3 \sin^4 \phi')$$

$$S_{\omega H} = (\sin^2 \phi' - \frac{1}{5})$$

$$S_{\omega D} = (\frac{3}{7} - \sin^2 \phi')$$

$$C_R = \left(\frac{A}{R}\right)^2 \left[J S_J + H \left(\frac{A}{R}\right) \sin \phi' S_H + 3 D \left(\frac{A}{R}\right)^2 S_D\right],$$

$$C_\omega = \left(\frac{A}{R}\right)^2 \left[2 J \sin \phi' + 3 H \left(\frac{A}{R}\right) S_{\omega H} + 4 D \left(\frac{A}{R}\right)^2 \sin \phi' S_{\omega D}\right].$$

it follows that

$$\ddot{\overline{R}}_g = - \frac{GM}{R^2} [(1 + C_R) \hat{R} + C_\omega \hat{\omega}].$$

B. THE CONTROL VARIABLE DIFFERENTIAL EQUATIONS

As was mentioned previously, a calculus of variations treatment of the problem of interest results in a system of differential equations to be satisfied by the control variables. These can be expressed as follows (Ref. 16):

$$\ddot{\overline{\lambda}} = (\overline{\lambda} \cdot \nabla) \ddot{\overline{R}}_g = \lambda_1 \frac{\partial \ddot{\overline{R}}_g}{\partial x} + \lambda_2 \frac{\partial \ddot{\overline{R}}_g}{\partial y} + \lambda_3 \frac{\partial \ddot{\overline{R}}_g}{\partial z}$$

Now, $\ddot{R}_{gi} = - \frac{GM}{R^2} \left[(1 + C_R) \frac{X_i}{R} + C_\omega \omega_i\right]$, where for $i = 1$, $\ddot{R}_{gi} = \ddot{X}_g$, etc.

$$\frac{\partial \ddot{R}_g}{\partial X_i} = -\left\{ \frac{\partial}{\partial X_i}\left(\frac{GM}{R^2}\right)\left[(1 + C_R)\,\hat{R} + C_\omega\,\hat{\omega}\right] + \frac{GM}{R^2}\left[\frac{\partial C_R}{\partial X_i}\,\hat{R}\right.\right.$$

$$\left.\left. + \frac{\partial C_\omega}{\partial X_i}\,\hat{\omega} + (1 + C_R)\,\frac{\partial}{\partial X_i}\,\hat{R}\right]\right\}$$

Taking the partials as indicated in each term, one finds that

$$\frac{\partial}{\partial X_i}\left(\frac{GM}{R^2}\right) = -\left(\frac{2\,GM}{R^3}\right)\frac{X_i}{R}$$

$$C_R = \left(\frac{A}{R}\right)^2\left[J\,S_J + H\left(\frac{A}{R}\right)\sin\phi'\,S_H + 3\,D\left(\frac{A}{R}\right)^2 S_D\right]$$

$$\frac{\partial C_R}{\partial X_i} = \frac{\partial}{\partial X_i}\left(\frac{A}{R}\right)^2(J\,S_J) + \frac{\partial}{\partial X_i}\left(\frac{A}{R}\right)^3(H\,S_H\sin\phi') + \frac{\partial}{\partial X_i}\left(\frac{A}{R}\right)^4(3\,D\,S_D)$$

$$+\left(\frac{A}{R}\right)^2\left[J\,\frac{\partial S_J}{\partial X_i} + H\left(\frac{A}{R}\right)\left(\frac{\partial(\sin\phi')}{\partial X_i}\,S_H + \sin\phi'\,\frac{\partial S_H}{\partial X_i}\right) + 3\,D\left(\frac{A}{R}\right)^2\frac{\partial S_D}{\partial X_i}\right]$$

$$\frac{\partial}{\partial X_i}\left(\frac{A}{R}\right)^n = -\frac{n}{R}\left(\frac{A}{R}\right)^n\frac{X_i}{R}$$

$$\frac{\partial S_J}{\partial X_i} = \frac{\partial}{\partial X_i}\left(1 - 5\sin^2\phi'\right) = \frac{-5.2}{R}\left(\sin\phi'\,\omega_i - \sin^2\phi'\,\frac{X_i}{R}\right)$$

$$\frac{\partial \sin\phi'}{\partial X_i} = \frac{1}{R}\left(\omega_i - \sin\phi'\,\frac{X_i}{R}\right)$$

$$\frac{\partial S_H}{\partial X_i} = \frac{\partial}{\partial X_i}\left(3 - 7\sin^2\phi'\right) = \frac{-7.2}{R}\left(\sin\phi'\,\omega_i - \sin^2\phi'\,\frac{X_i}{R}\right)$$

$$\frac{\partial S_D}{\partial X_i} = \frac{\partial}{\partial X_i}\left(\frac{1}{7} - 2\sin^2\phi' + 3\sin^4\phi'\right) = \frac{-2.2}{R}\left(\sin\phi'\,\omega_i - \sin^2\phi'\,\frac{X_i}{R}\right)$$

$$+ \frac{3\cdot 4}{R}\left(\sin^3\phi'\,\omega_i - \sin^4\phi'\,\frac{X_i}{R}\right)$$

$$\frac{\partial C_R}{\partial X_i} = -\left[\frac{2\,J}{R}\left(\frac{A}{R}\right)^2 S_J + \frac{3\,H}{R}\left(\frac{A}{R}\right)^3 S_H \sin\phi' + \frac{12\,D}{R}\left(\frac{A}{R}\right)^4 S_D\right]\frac{X_i}{R}$$

$$+ \left(\frac{A}{R}\right)^2\left[-\frac{10\,J}{R}\sin\phi' + H\left(\frac{A}{R}\right)\left(\frac{1}{R}S_H - \frac{14}{R}\sin^2\phi'\right)\right.$$

$$\left.+\, 3\,D\left(\frac{A}{R}\right)^2\left(\frac{-4}{R}\sin\phi' + \frac{12}{R}\sin^3\phi'\right)\right]\omega_i + \left(\frac{A}{R}\right)^2\left[+\frac{10\,J}{R}\sin^2\phi'\right.$$

$$\left.+\, H\left(\frac{A}{R}\right)\left(-\frac{\sin\phi'}{R}S_H + \frac{14}{R}\sin^3\phi'\right) + 3\,D\left(\frac{A}{R}\right)^2\left(\frac{4}{R}\sin^2\phi' - \frac{12}{R}\sin^4\phi'\right)\right]\frac{X_i}{R}$$

$$C_\omega = \left(\frac{A}{R}\right)^2\left[2\,J\sin\phi' + 3\,H\left(\frac{A}{R}\right)S_{\omega H} + 4\,D\left(\frac{A}{R}\right)^2\sin\phi'\,S_{\omega D}\right]$$

$$\frac{\partial C_\omega}{\partial X_i} = \left[\frac{\partial}{\partial X_i}\left(\frac{A}{R}\right)^2(2\,J\sin\phi') + \frac{\partial}{\partial X_i}\left(\frac{A}{R}\right)^3(3\,H\,S_{\omega H})\right.$$

$$\left.+\,\frac{\partial}{\partial X_i}\left(\frac{A}{R}\right)^4(4\,D\sin\phi'\,S_{\omega D})\right] + \left(\frac{A}{R}\right)^2\left[2\,J\,\frac{\partial(\sin\phi')}{\partial X_i}\right.$$

$$\left.+\,3\,H\left(\frac{A}{R}\right)\frac{\partial S_{\omega H}}{\partial X_i} + 4\,D\left(\frac{A}{R}\right)^2\left(\frac{\partial\sin\phi'}{\partial X_i}S_{\omega D} + \sin\phi'\,\frac{\partial S_{\omega D}}{\partial X_i}\right)\right]$$

$$\frac{\partial\,(\sin\phi')}{\partial X_i} = \frac{1}{R}\,(\,\omega_i - \sin\phi'\,\frac{X_i}{R}\,)$$

$$\frac{\partial\, S_{\omega H}}{\partial X_i} = \frac{\partial}{\partial X_i}\,(\sin^2\phi' - \frac{1}{5}) = \frac{2}{R}\,(\sin\phi'\,\omega_i - \sin^2\phi'\,\frac{X_i}{R}\,)$$

$$\frac{\partial\, S_{\omega H}}{\partial X_i} = \frac{\partial}{\partial X_i}\,(\sin^2\phi' - \frac{1}{5}\,) = \frac{2}{R}\,(\sin\phi'\,\omega_i - \sin^2\phi'\,\frac{X_i}{R}\,)$$

$$\frac{\partial S_{\omega D}}{\partial X_i} = \frac{\partial}{\partial X_i}\,(\frac{3}{7} - \sin^2\phi') = \frac{-2}{R}\,(\sin\phi'\,\omega_i - \sin^2\phi'\,\frac{X_i}{R}\,)$$

$$\frac{\partial C_\omega}{\partial X_i} = -\left[\frac{4\,J}{R}\left(\frac{A}{R}\right)^2\sin\phi' + \frac{9\,H}{R}\left(\frac{A}{R}\right)^3 S_{\omega H} + \frac{16\,D}{R}\left(\frac{A}{R}\right)^4 \sin\phi'\,S_{\omega D}\right]\frac{X_i}{R}$$

$$+\left(\frac{A}{R}\right)^2\left[\frac{2\,J}{R} + \frac{6\,H}{R}\left(\frac{A}{R}\right)\sin\phi' + 4\,D\left(\frac{A}{R}\right)^2\left(\frac{S_{\omega D}}{R} - \frac{2}{R}\sin^2\phi'\right)\right]\omega_i$$

$$+\left(\frac{A}{R}\right)^2\left[-\frac{2\,J}{R}\sin\phi' - \frac{6\,H}{R}\left(\frac{A}{R}\right)\sin^2\phi' + 4\,D\left(\frac{A}{R}\right)^2\left(-\frac{\sin\phi'}{R}\,S_{\omega D}\right.\right.$$

$$\left.\left. + \frac{2}{R}\sin^3\phi'\right)\right]\frac{X_i}{R}$$

$$\frac{\partial}{\partial X_i}\left(\frac{X}{R}\right) = \frac{\delta_{1i}}{R} - \frac{XX_i}{R^3} = \frac{1}{R}\left(\delta_{1i} - \frac{X}{R}\,\frac{X_i}{R}\right) \quad \text{where } \delta_{1i} = 1,\ 0,\ 0 \quad \text{as} \quad i = 1,\ 2,\ 3$$

There are only two more partials to be established to complete the $\ddot{\lambda}$ expression; they are as follows:

$$\frac{\partial}{\partial X_i}\left(\frac{Y}{R}\right) = \frac{1}{R}\left(\delta_{2i} - \frac{Y}{R}\,\frac{X_i}{R}\right), \quad \delta_{2i} = 0,\ 1,\ 0 \quad \text{as} \quad i = 1,\ 2,\ 3,$$

$$\frac{\partial}{\partial X_i}\left(\frac{Z}{R}\right) = \frac{1}{R}\left(\delta_{3i} - \frac{Z}{R}\,\frac{X_i}{R}\right), \quad \delta_{3i} = 0,\ 0,\ 1 \quad \text{as} \quad i = 1,\ 2,\ 3.$$

Now one may write $\frac{\partial \ddot{R}_g}{\partial X_i}$ explicitly as follows:

$$\frac{\partial \ddot{R}_g}{\partial X_i} = \left(\frac{2\,GM}{R^3}\right) \frac{X_i}{R} \left[(1 + C_R)\, \hat{R} + C_\omega\, \hat{\omega}\right] + \frac{GM}{R^2}\left[(1 + C_R)\frac{1}{R}\left(-\delta_{1i} + \hat{R}\,\frac{X_i}{R}\right)\right]$$

$$+ \frac{GM}{R^2}\left\{\left(\frac{A}{R}\right)^2\left[\frac{2\,J}{R}\, S_J + \frac{3\,H}{R}\left(\frac{A}{R}\right) S_H \sin\phi' + \frac{12\,D}{R}\left(\frac{A}{R}\right)^2 S_D\right.\right.$$

$$- \frac{10\,J}{R}\sin^2\phi' + \frac{H}{R}\left(\frac{A}{R}\right)\left(S_H \sin\phi' - \frac{14}{R}\sin^3\phi'\right) + \frac{3\,D}{R}\left(\frac{A}{R}\right)^2 (\,-4\sin^2\phi'$$

$$\left. + 12\sin^4\phi')\right] \frac{X_i}{R}\,\hat{R} + \left(\frac{A}{R}\right)^2\left[\frac{10\,J}{R}\sin\phi' - H\left(\frac{A}{R}\right)\left(\frac{1}{R}S_H - \frac{14}{R}\sin^2\phi'\right)\right.$$

$$\left. - 3\,D\left(\frac{A}{R}\right)^2\left(-\frac{4}{R}\sin\phi' + \frac{12}{R}\sin^3\phi'\right)\right]\omega_i\,\hat{R} + \left(\frac{A}{R}\right)^2\left[\frac{4\,J}{R}\sin\phi'\right.$$

$$+ \frac{9\,H}{R}\left(\frac{A}{R}\right) S_{\omega D} + \frac{16\,D}{R}\left(\frac{A}{R}\right)^2 \sin\phi'\, S_{\omega D} + \frac{2\,J}{R}\sin\phi' + \frac{6\,H}{R}\left(\frac{A}{R}\right)\sin^2\phi'$$

$$\left. + \frac{4\,D}{R}\left(\frac{A}{R}\right)^2 (\sin\phi'\, S_{\omega D} - 2\sin^3\phi')\right]\frac{X_i}{R}\,\hat{\omega} + \left(\frac{A}{R}\right)^2\left[-\frac{2\,J}{R}\right.$$

$$\left.\left. - \frac{6\,H}{R}\left(\frac{A}{R}\right)\sin\phi' - \frac{4\,D}{R}\left(\frac{A}{R}\right)^2 (2\sin^2\phi' - S_{\omega D})\right]\omega_i\,\hat{\omega}\right\}$$

From the above explicit form, one can generalize to $\ddot{\lambda}$ without writing all the other details. Introducing some new notation, one may write the following:

$$\sigma_{R\omega} = \left[10\,J\sin\phi' - H\left(\frac{A}{R}\right)(S_H - 14\sin^2\phi') - 12\,D\left(\frac{A}{R}\right)^2 (3\sin^2\phi' - 1)\sin\phi'\right]$$

$$\sigma_{\omega\omega} = -\left[2\,J + 6\,H\left(\frac{A}{R}\right)\sin\phi' - 4\,D\left(\frac{A}{R}\right)^2 (2\sin^2\phi' - S_{\omega D})\right]$$

$$\ddot{\bar{\lambda}} = \frac{GM}{R^3} \{ (\bar{\lambda} \cdot \hat{R}) [3 (1 + C_R) \hat{R} + 2 C_\omega \hat{\omega}] - (1 + C_R) \bar{\lambda} \}$$

$$+ \frac{GM}{R^3} \left(\frac{A}{R}\right)^2 \left\{ \left\{ (\bar{\lambda} \cdot \hat{R}) \left[2 J S_J + 3 H \left(\frac{A}{R}\right) S_H \sin\phi' + 12 D \left(\frac{A}{R}\right)^2 S_D - \sigma_{R\omega} \sin\phi' \right] + (\bar{\lambda} \cdot \hat{\omega}) \sigma_{R\omega} \right\} \hat{R} \right.$$

$$\left. + \left\{ (\bar{\lambda} \cdot \hat{R}) \left[4 J \sin\phi' + 9 H \left(\frac{A}{R}\right) S_{\omega H} + 16 D \left(\frac{A}{R}\right)^2 \sin\phi' S_{\omega D} - \sigma_{\omega\omega} \sin\phi' \right] + (\bar{\lambda} \cdot \hat{\omega}) \sigma_{\omega\omega} \right\} \hat{\omega} \right\}$$

At this point one can see that by computing the dot product

$$(\bar{\lambda} \cdot \hat{\omega})$$

and using the coefficients

$$(1 + C_R) \text{ and } C_\omega$$

from the $\ddot{\bar{R}}_g$ expression, the dominant part of the $\ddot{\bar{\lambda}}$ expression can be evaluated as cheaply as the spherical part. In Reference 3, the $\ddot{\bar{\lambda}}$ expression for a spherical earth should be given as

$$\ddot{\bar{\lambda}}_S = \frac{GM}{R^3} [3 (\bar{\lambda} \cdot \hat{R}) \hat{R} - \bar{\lambda}].$$

Defining the expression derived here to consist of a "dominant" and "minor" part indicated by

$\ddot{\bar{\lambda}} = \ddot{\bar{\lambda}}_D + \ddot{\bar{\lambda}}_M$, it is seen that

$$\ddot{\bar{\lambda}}_D = \frac{GM}{R^3} \{ (\bar{\lambda} \cdot \hat{R}) [3 (1 + C_R) \hat{R} + 2 C_\omega \hat{\omega}] - (1 + C_R) \bar{\lambda} \}$$

and

$$\ddot{\bar{\lambda}}_M = \frac{GM}{R^3}\left(\frac{A}{R}\right)^2 \left\{\left\{(\bar{\lambda} \cdot \hat{R})\left[2\,J\,S_J + 3\,H\left(\frac{A}{R}\right) S_H \sin\phi' + 12\,D\left(\frac{A}{R}\right)^2 S_D - \sigma_{R\omega}\sin\phi'\right] + (\bar{\lambda}\cdot\hat{\omega})\,\sigma_{R\omega}\right\}\hat{R} + \left\{(\bar{\lambda}\cdot\hat{R})\left[4\,J\sin\phi' + 9\,H\left(\frac{A}{R}\right) S_{\omega H} + 16\,D\left(\frac{A}{R}\right)^2 S_{\omega D}\sin\phi' - \sigma_{\omega\omega}\sin\phi'\right] + (\bar{\lambda}\cdot\hat{\omega})\,\sigma_{\omega\omega}\right\}\hat{\omega}\right\}$$

where

$$\sigma_{R\omega} = 10\,J\sin\phi' - H\left(\frac{A}{R}\right)(S_H - 14\sin^2\phi') - 12\,D\left(\frac{A}{R}\right)^2 (3\sin^2\phi' - 1)\sin\phi'$$

$$\sigma_{\omega\omega} = -\left[2\,J + 6\,H\left(\frac{A}{R}\right)\sin\phi' - 4\,D\left(\frac{A}{R}\right)^2 (2\sin^2\phi' - S_{\omega D})\right].$$

SECTION IV. EFFICIENT USE OF PARAMETERS REPRESENTING THE CONTROL VARIABLES

Assume that the basic system of equations (the mathematical representation of the problem) is already developed in a space-fixed cartesian coordinate system. The objective here is to determine efficient techniques for using this development rather than to investigate whether it should have been in space-fixed spherical coordinates, rotating cartesian coordinates, etc. Many computer programs are already in use based on the space-fixed cartesian coordinate development.

The control variables can be represented by various sets of parameters, depending upon the reference coordinate system, the choice of variables, etc. It goes without saying that each set of parameters has certain advantages or disadvantages for performing various tasks, or for solving various problems.

Since the differential equations are developed in terms of $\bar{\lambda}$, $\dot{\bar{\lambda}}$, and $\ddot{\bar{\lambda}}$, there is a tendency to specify initial conditions in terms of the "LAMBDAS," i.e. $\bar{\lambda}_o$ and $\dot{\bar{\lambda}}_o$. The transformations developed here allow the user to consider three sets of parameters rather than just the "LAMBDAS." An analysis is included to point out advantages and disadvantages associated with the various sets of parameters, and an efficient procedure is given for their use in solving the two-point boundary value problem.

A. DISCUSSION AND ILLUSTRATION OF PARAMETERS REPRESENTING THE CONTROL VARIABLES

Recall now that the control variables have the following physical interpretation [Ref. 4]:

$\bar{\lambda} = \Lambda \hat{\lambda}$: A vector of magnitude Λ along the thrust direction, $\hat{\lambda}$.

$\dot{\bar{\lambda}} = \Lambda \dot{\hat{\lambda}} + \dot{\Lambda} \hat{\lambda}$: Time rate of change of $\bar{\lambda}$.

Notice in Figure 1 that the thrust direction, $\hat{\lambda}$, only gives two independent control parameters. These are depicted in the figure by the angles χ_p and χ_y. A first impression might be that $\bar{\lambda}$ offers three independent parameters, but an examination of the differential equations (Ref. 3) reveals that Λ has no influence on the trajectory.

Figure 1 also illustrates two ways in which one may specify initial conditions for the control variables. One way is to specify $\bar{\lambda}_0$ and $\dot{\bar{\lambda}}_0$. Another is to specify χ_p, χ_y, $\dot{\chi}_p$, $\dot{\chi}_y$, Λ, and $\dot{\Lambda}$.

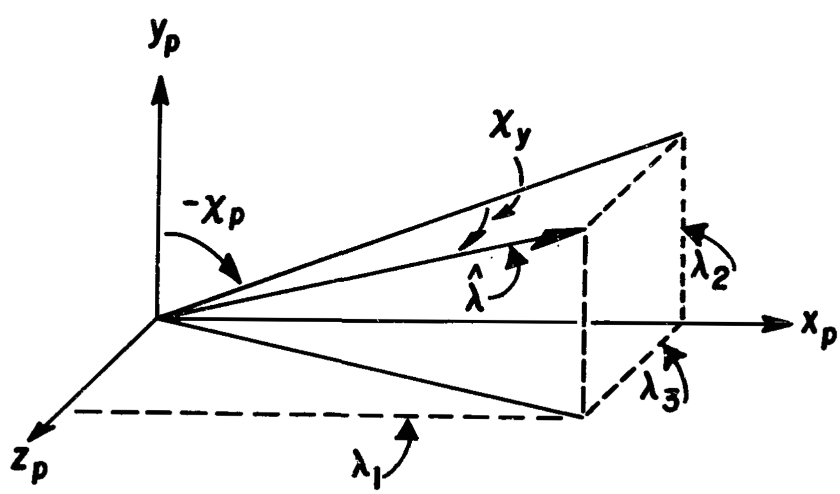

FIGURE 1. CONTROL VARIABLES RELATIVE TO A CARTESIAN SYSTEM

To clarify what is meant here, consider the transformation from the control parameters $(\chi_p, \chi_y, \dot{\chi}_p, \dot{\chi}_y, \Lambda, \dot{\Lambda})$ to the control parameters $(\bar{\lambda}, \dot{\bar{\lambda}})$. From Figure 1 it is seen that the unit thrust vector is expressed in terms of χ_p and χ_y as

$$\hat{\lambda} = \begin{pmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \end{pmatrix} = \begin{pmatrix} -\cos \chi_y \sin \chi_p \\ \cos \chi_y \cos \chi_p \\ \sin \chi_y \end{pmatrix}$$

Differentiating with respect to time, it follows that

$$\dot{\hat{\lambda}} = \begin{pmatrix} \sin \chi_y \sin \chi_p \dot{\chi}_y - \cos \chi_y \cos \chi_p \dot{\chi}_p \\ -\sin \chi_y \cos \chi_p \dot{\chi}_y - \cos \chi_y \sin \chi_p \dot{\chi}_p \\ \cos \chi_y \dot{\chi}_y \end{pmatrix}$$

To complete the transformation, note the relationship

$\bar{\lambda} = \Lambda \hat{\lambda}$, and differentiate to get $\dot{\bar{\lambda}} = \Lambda \dot{\hat{\lambda}} + \dot{\Lambda} \hat{\lambda}$. It is seen then that the "CHIS," $(\chi_p, \chi_y, \dot{\chi}_p, \dot{\chi}_y, \Lambda, \dot{\Lambda})$, represent a set of control parameters, just as do the "LAMBDAS," $(\bar{\lambda}, \dot{\bar{\lambda}})$.

Now, getting back to a primary question: "What are the advantages or disadvantages of specifying initial values in one set of parameters rather than the other?"

One important point may be stated as follows: "Specification of parameters referenced to an arbitrary coordinate system is likely to result in unnecessary difficulties."

Both the "CHI" and "LAMBDA" parameters are associated with an arbitrary coordinate system. The unnecessary difficulties associated with their specification is illustrated in Figure 2.

Figure 2 illustrates a current problem of interest, that of leaving a circular parking orbit of radius R_C and burning into an elliptic orbit. Assume now that the distance from the origin to M_1 is equal to the distance from origin to M_2, and that the flight time from injection to M_1 is equal to that from injection to M_2. This is essentially a simplification of the lunar injection problem where M_1 represents the moon on one day, and M_2 represents the moon on some other day.

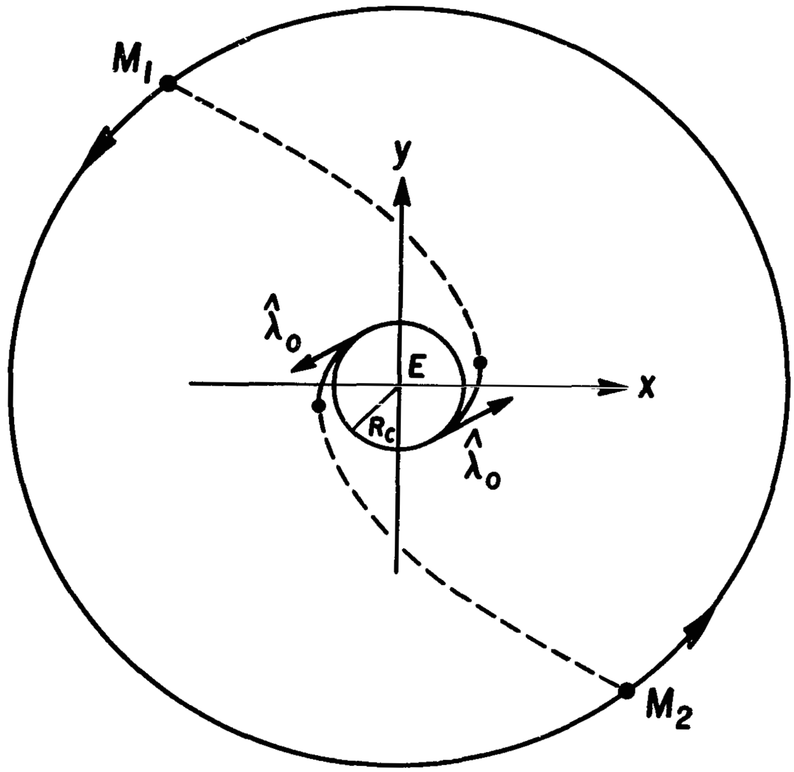

FIGURE 2. CONTROL VARIABLE VECTORS FOR A PHYSICAL PROBLEM

The point to be made is that really the physical problem has not changed in the two situations, but the control parameters shown in the figure are completely different. Figure 3 illustrates the difference, which in this simple illustration can be remedied rather easily. Unfortunately, in most problems such difficulties waste many hours of computer time.

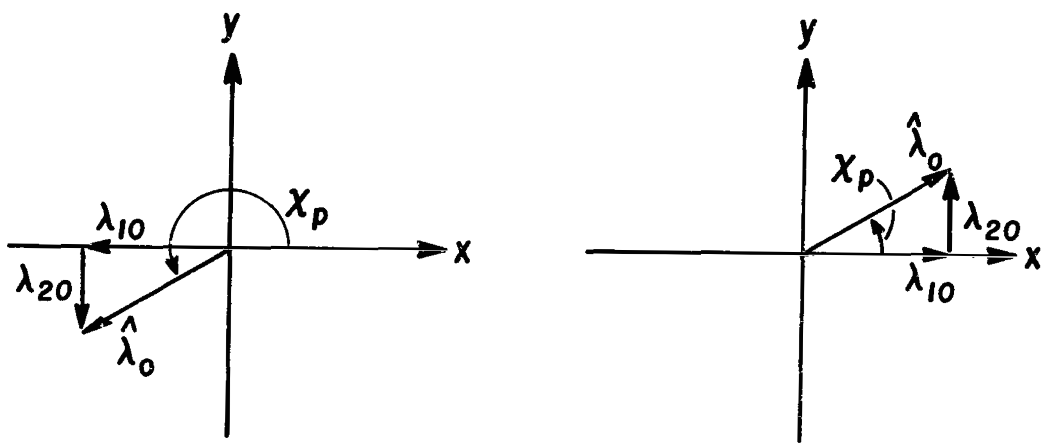

FIGURE 3. COMPONENTS OF THE VARIED CONTROL VECTOR

Now consider a set of parameters that eliminate these unnecessary difficulties. Call this set the "ALPHA" parameters. They have a three dimensional physical basis rather than an arbitrary coordinate system basis.

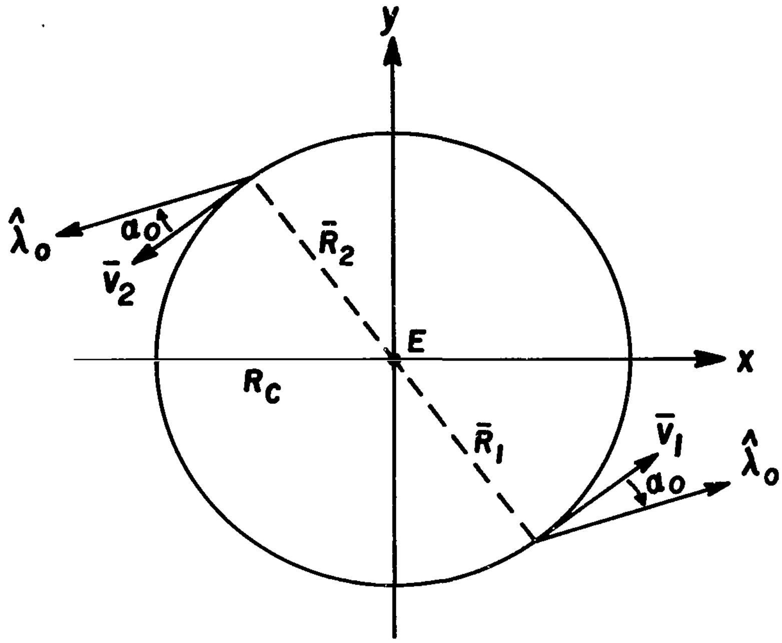

FIGURE 4. CONTROL VARIABLES RELATIVE TO A FUNDAMENTAL SYSTEM

Figure 4 illustrates the inplane significance of the "ALPHA" parameters for the same problem as discussed previously. Notice that the "α_0" for going to M_1 is the same as the "α_0" for going to M_2 (under the previous problem definition). Thus, the "ALPHA" parameters are problem oriented. Having solved the physical problem once, the parameters remain unchanged regardless of coordinate system definition, or any other changes that do not change the fundamental problem.

The transformations relating the "CHI", "LAMBDA", and "ALPHA" parameters are developed in detail in the next part of this report. The transformations are developed in three dimensions using vector notation as defined previously.

Some experimentation has been done to determine how one may use the various sets of parameters to his advantage. The results can be summarized as follows:

1. The "ALPHA" parameters (α_n, α_w, $\dot{\alpha}_n$, $\dot{\alpha}_w$, Λ, $\dot{\Lambda}$) are more efficient than either the "CHIS" or "LAMBDAS" for specifying initial conditions, as was pointed out in the preceding figures and associated discussions. When one specifies "LAMBDAS" he is in effect specifying direction cosines and their rate of change with respect to an arbitrarily oriented coordinate system. This is a difficult and unnecessary task.

2. "LAMBDA" parameters are more efficient than "ALPHAS" or "CHIS" for use in multivariable isolation schemes. This is easy to visualize by thinking of orbital inclination as a desired end-condition and an inplane alpha as a control parameter. The influence coefficient (partial derivative) of orbital inclination with respect to an inplane alpha is zero, i.e. $\frac{\partial I}{\partial \alpha_n} = 0$. This type of thing leads to difficulties in the use of multivariable isolation schemes. The use of "LAMBDAS" avoids such difficulties.

In summary, "ALPHAS" are used to specify initial values for the control variables. These initial "ALPHAS" are then transformed to "LAMBDAS" for use in the multivariable isolation scheme to achieve desired end conditions. The inverse transformation, i.e. the "LAMBDA" to "ALPHA" transformation, yeilds the "ALPHAS" associated with any set of "LAMBDAS". These transformations allow the user to utilize the "ALPHAS" and "LAMBDAS" to his advantage rather than being forced to use one or the other in an uneconomical fashion.

B. TRANSFORMATIONS RELATING THE PARAMETER SETS, THE "CHIS", "ALPHAS", AND "LAMBDAS"

These transformations are valid with respect to any space-fixed cartesian coordinate system. The "CHIS" carry the relationship depicted in Figure 1 with respect to the coordinate axes. Obviously, the quantities defining position, velocity, etc. must also be taken relative to the same coordinate system.

1. Transforming "LAMBDAS", $(\bar{\lambda}, \dot{\bar{\lambda}})$, to "CHIS", $(\chi_y, \chi_p, \dot{\chi}_y, \dot{\chi}_p, \Lambda$, and $\dot{\Lambda})$

$\bar{\lambda}$; vector defining the direction of the thrust.

$\dot{\bar{\lambda}}$; vector defining the time derivative of the thrust direction.

$\Lambda^2 = \bar{\lambda} \cdot \bar{\lambda}$; square of the magnitude of $\bar{\lambda}$, i.e. $\Lambda = |\bar{\lambda}|$

$$\hat{\lambda} = \begin{pmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \end{pmatrix} = \bar{\lambda}/\Lambda \text{ ; unit vector in the thrust direction}$$

From Figure 1, one can show that

$$\sin \chi_y = \lambda_3'$$

$$\cos \chi_y = (\lambda_1'^2 + \lambda_2'^2)^{\frac{1}{2}}$$

(equations for evaluating the angles χ_y, and χ_p).

$$\sin \chi_p = -\lambda_1'/\cos \chi_y$$

$$\cos \chi_p = \lambda_2'/\cos \chi_y$$

Differentiating with respect to time yields

$$\dot{\chi}_y = \frac{1}{\Lambda \cos \chi_y} \left(\dot{\lambda}_3' - \dot{\Lambda} \sin \chi_y \right)$$

$$\dot{\chi}_p = \frac{1}{\lambda_1'} \left[\dot{\lambda}_2' - \lambda_2' \left(\frac{\lambda_1' \dot{\lambda}_1' + \lambda_2' \dot{\lambda}_2'}{\lambda_1'^2 + \lambda_2'^2} \right) \right],$$

(Equations for computing the time rate of change of the angles χ_y and χ_p, once the vector $\dot{\hat{\lambda}}$ is computed.)

To compute $\dot{\hat{\lambda}}$ from the known $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ vectors, recall that

$$\bar{\lambda} = \Lambda \hat{\lambda}$$

Differentiation gives

$$\dot{\bar{\lambda}} = \Lambda \dot{\hat{\lambda}} + \dot{\Lambda} \lambda .$$

Since $\Lambda \dot{\Lambda} = \bar{\lambda} \cdot \dot{\bar{\lambda}}$, the expression

$$\dot{\hat{\lambda}} = \frac{1}{\Lambda} \left(\dot{\bar{\lambda}} - \dot{\Lambda} \hat{\lambda} \right)$$ can be evaluated.

2. Transforming "CHIS," (χ_y, χ_p, $\dot{\chi}_y$, $\dot{\chi}_p$, $\Lambda, \dot{\Lambda}$), to "LAMBDAS," ($\bar{\lambda}$, $\dot{\bar{\lambda}}$)

$$\hat{\lambda} = \begin{pmatrix} -\cos \chi_y \sin \chi_p \\ \cos \chi_y \cos \chi_p \\ \sin \chi_y \end{pmatrix}$$; unit vector in the direction of the thrust vector.

$$\bar{\lambda} = \Lambda \hat{\lambda}$$; vector of magnitude Λ in the thrust direction

$$\dot{\hat{\lambda}} = \begin{pmatrix} \sin \chi_y \sin \chi_p \dot{\chi}_y - \cos \chi_y \cos \chi_p \dot{\chi}_p \\ - \sin \chi_y \cos \chi_p \dot{\chi}_y - \cos \chi_y \sin \chi_p \dot{\chi}_p \\ \cos \chi_y \dot{\chi}_y \end{pmatrix}$$

$$\dot{\bar{\lambda}} = \Lambda \dot{\hat{\lambda}} + \dot{\Lambda} \hat{\lambda}$$

3. Transforming "ALPHAS," (α_n, α_w, $\dot{\alpha}_n$, $\dot{\alpha}_w$, Λ, $\dot{\Lambda}$), to "LAMBDAS," ($\bar{\lambda}$, $\dot{\bar{\lambda}}$)

$\bar{R}$, $\bar{V}$, $\bar{a}$ represent vehicle position, velocity, and acceleration relative to the same coordinate system in which $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ are desired.

a. Computation of some basic quantities

$R^2 = \bar{R} \cdot \bar{R}$

$V^2 = \bar{V} \cdot \bar{V}$

$R \dot{R} = \bar{R} \cdot \bar{V}$

$C_1 = \sqrt{R^2 V^2 - (R \dot{R})^2}$; angular momentum magnitude

$\bar{W} = \bar{R} \times \bar{V}$; angular momentum vector

$\hat{R} = \bar{R}/R$; unit position vector

$\hat{V} = \bar{V}/V$; unit velocity vector

$\hat{W} = \bar{W}/C_1$; unit vector in the angular momentum direction; unit normal to the flight plane.

$\hat{N} = \hat{W} \times \hat{V}$; auxiliary unit vector to complete the orthogonal right-hand coordinate system $\hat{V}\ \hat{N}\ \hat{W}$.

b. Quantities depending on the "ALPHAS"

$\rho_v = \cos \alpha_n \cos \alpha_w$

$\rho_n = \sin \alpha_n \cos \alpha_w$; direction cosines of the $\hat{\lambda}$ on the axes $\hat{V}$, $\hat{N}$, $\hat{W}$.

$\rho_w = \sin \alpha_w$

$\dot{\rho}_v = - (\rho_n \dot{\alpha}_n + \rho_w \cos \alpha_n \dot{\alpha}_w)$

$\dot{\rho}_n = \rho_v \dot{\alpha}_n - \rho_w \sin \alpha_n \dot{\alpha}_w$

$\dot{\rho}_w = \cos \alpha_w \dot{\alpha}_w$

$\hat{\lambda} = \rho_v \hat{V} + \rho_n \hat{N} + \rho_w \hat{W}$; unit vector in the thrust direction.

$\bar{\lambda} = \Lambda \hat{\lambda}$; this computation is not necessary since the trajectory is unchanged whether $\hat{\lambda}$ or $\bar{\lambda}$ is used.

$\bar{a} = \frac{F}{M} \hat{\lambda} + \ddot{\bar{R}}_g$; acceleration vector where $\ddot{\bar{R}}_g$ represents the acceleration due to gravity.

$V \dot{V} = \bar{V} \cdot \bar{a}$

$\dot{C}_1 = \frac{1}{C_1} [R^2 (V \dot{V}) - R \dot{R} (\bar{R} \cdot \bar{a})]$; time rate of change of C_1

$\dot{\hat{V}} = \frac{1}{V} (\bar{a} - \dot{V} \hat{V})$; time rate of change of the vectors $\hat{V}$, $\hat{W}$, and $\hat{N}$.

$\dot{\hat{W}} = \frac{1}{C_1} [(\bar{R} \times \bar{a}) - \dot{C}_1 \hat{W}]$

$\dot{\hat{N}} = (\dot{\hat{W}} \times \hat{V}) + (\hat{W} \times \dot{\hat{V}})$

$\dot{\hat{\lambda}} = \dot{\rho}_v \hat{V} + \rho_v \dot{\hat{V}} + \dot{\rho}_n \hat{N} + \rho_n \dot{\hat{N}} + \dot{\rho}_w \hat{W} + \rho_w \dot{\hat{W}}$

$\dot{\bar{\lambda}} = \Lambda \dot{\hat{\lambda}} + \dot{\Lambda} \hat{\lambda}$; time rate of change of the thrust vector direction

4. Transforming "LAMBDAS," $(\bar{\lambda}, \dot{\bar{\lambda}})$, to "ALPHAS," $(\alpha_n, \alpha_w, \dot{\alpha}_n, \dot{\alpha}_w, \Lambda, \dot{\Lambda})$

Some terms defined in the previous transformation are used here without redefining them or rewriting the equations.

$$\left.\begin{aligned} \Lambda^2 &= \bar{\lambda} \cdot \bar{\lambda} \\ \Lambda \dot{\Lambda} &= \bar{\lambda} \cdot \dot{\bar{\lambda}} \end{aligned}\right\}$$ these equations yield the two parameters Λ and $\dot{\Lambda}$ associated with the current $\bar{\lambda}$ and $\dot{\bar{\lambda}}$.

$\hat{\lambda} = \bar{\lambda}/\Lambda$; unit thrust vector

$$\left.\begin{aligned} \rho_v &= \hat{\lambda} \cdot \hat{V} \\ \rho_n &= \hat{\lambda} \cdot \hat{N} \\ \rho_w &= \hat{\lambda} \cdot \hat{W} \end{aligned}\right\}$$ direction cosines of the unit thrust vector relative to the $\hat{V}$, $\hat{N}$, and $\hat{W}$ axes.

$$\left.\begin{aligned} \sin \alpha_w &= \rho_w \\ \cos \alpha_w &= \sqrt{\rho_v^2 + \rho_n^2} \\ \sin \alpha_n &= \rho_n / \cos \alpha_w \\ \cos \alpha_n &= \rho_v / \cos \alpha_w \end{aligned}\right\}$$ these equations yield the parameters α_n and α_w.

$$\dot{\hat{\lambda}} = \frac{1}{\Lambda} (\dot{\bar{\lambda}} - \dot{\Lambda} \hat{\lambda})$$

$$\dot{\rho}_v = \dot{\hat{\lambda}} \cdot \hat{V} + \hat{\lambda} \cdot \dot{\hat{V}}$$

$$\dot{\rho}_n = \dot{\hat{\lambda}} \cdot \hat{N} + \hat{\lambda} \cdot \dot{\hat{N}}$$

$$\dot{\rho}_w = \dot{\hat{\lambda}} \cdot \hat{W} + \hat{\lambda} \cdot \dot{\hat{W}}$$

$$\dot{\alpha}_w = \dot{\rho}_w / \cos \alpha_w$$

$$\dot{\alpha}_n = \frac{1}{\rho_v} (\dot{\rho}_n + \rho_w \sin \alpha_n \dot{\alpha}_w)$$

C. APPLICATION OF THE RECOMMENDED PARAMETERS TO GENERATE TIME OPTIMAL LUNAR TRAJECTORIES

An IBM 7094 program is in operation using these parameters as recommended to generate time optimal lunar trajectories. To illustrate the advantages gained, consider the program option where injection C_3 (twice the total energy per unit mass) is the only terminal constraint on the trajectory burn phase from earth parking orbit to lunar transit injection; this option uses booster liftoff time, earth parking orbit coast time, and the burn time (equivalent to C_3 at injection, C_{3i}) as parameters to isolate three desired lunar arrival conditions.

The recommended procedure is that the initial values for the control parameters be given in terms of the "ALPHAS," i.e. (α_n, α_w, $\dot{\alpha}_n$, $\dot{\alpha}_w$, Λ, $\dot{\Lambda}$). Now assume no information is available for specifying initial conditions for the "ALPHAS." By using the initial values

$$\alpha_n = \alpha_w = \dot{\alpha}_n = \dot{\alpha}_w = 0$$

$$\Lambda = 1$$

$$\dot{\Lambda} = 0$$

the program, using the associated "LAMBDAS," will isolate any realistic required C_{3i}, and the five associated transversality conditions. Usually eight to twenty-five powered flight phases are required to obtain this optimum set of control parameters. Having found the required "LAMBDAS" to achieve the desired C_{3i} and transversality conditions, the associated "ALPHAS" are carried from one trajectory to the other as launch time, coast time, and burn time are varied to arrive at desired periselenum conditions. This means that once the optimum set of initial "ALPHAS" are determined for the set of desired terminal conditions of the burn phase they are essentially invariant as launch time, coast time, and burn time are varied to achieve desired periselenum conditions. Probably, if large variations in burn time were required, the initial "ALPHAS" would have to be determined again to stay within some preset tolerance of the exact optimum values.

The procedures discussed here are valid regardless of the orientation of the cartesian coordinate system being used, or the position of the moon in its orbit. They hold for the variations normally made in launch time, coast time, and burn time to arrive at desired periselenum conditions, with only one search to determine optimum initial values for the "ALPHAS." This is possible by specifying initial values for the shaping parameters in terms of the "ALPHAS" and utilizing the transformations that relate the "ALPHAS" and "LAMBDAS."

D. CONCLUSIONS AS TO PARAMETERS REPRESENTING THE CONTROL VARIABLES

There is a best set of parameters for analyzing a given problem. To some extent it is uneconomical to analyze in terms of any parameters other than that best set. Determining the best set may be practically impossible, but efforts to that end are normally worthwhile.

For the problem treated in this report, experience has shown the "ALPHA" parameters to be better for specifying initial conditions for the control variables. The "LAMBDA" parameters are better for performing multivariable isolations to determine initial values for the control variables that produce the trajectory to desired terminal conditions. Hence, the transformations relating these parameters allow the user to take advantage of these properties. This was pointed out in the application of the techniques to generate time optimal lunar trajectories.

A survey of the various physical problems can be conducted using a simplified model to determine approximate initial values for the "ALPHAS." When the survey is completed, the resulting data can be represented functionally by using curve-fitting techniques. These functions can therefore be built into appropriate computer programs. This should relieve the user of the task of trying to guess what initial values of the control parameters are required for various vehicles to leave various parking orbits and burn into various terminal conic sections.

SECTION V. TERMINAL CONSTRAINTS AND TRANSVERSALITY CONDITIONS

The following development is applicable to problems having fixed initial state conditions but functionally variable terminal state conditions. The object is to transfer the vehicle from the initial state conditions to the desired terminal state conditions in a minimum time. This is accomplished by finally determing initial conditions for the control variables such that the terminal constraints are satisfied in the time optimal fashion.

It is possible to grasp and use the methods for solving this problem without going through the theoretical development of the methods [References 1, 2, 6, and 7, present the theoretical development]. The objective here is to present the methods in an understandable form, to develop a generalized set of transversality conditions for a particularly useful class of constraints, and to present in detail one application for the generalized transversality conditions. The constraints necessary to generate time optimal lunar or interplanetary trajectories from a parking orbit (assuming one burn phase from orbit to injection) are of the acceptable class, hence, these are used in the detailed application just mentioned.

A. FUNDAMENTAL DISCUSSION OF TRANSVERSALITY CONDITIONS

References 1, 2, 6, and 7 all give rules in one form or another such that for the problem at hand one can show that

$$\lambda_i = -\sum_{j=1}^{J} P_j \frac{\partial F_j}{\partial \dot{X}_i} \quad \text{and} \quad \dot{\lambda}_i = \sum_{j=1}^{J} P_j \frac{\partial F_j}{\partial X_i}$$

are relationships that must be satisfied at the terminal state surface (hypersurface) along the time optimal trajectory. The terms in these expressions are defined as follows:

F_j $(j=1, \ldots, J)$: Desired terminal state conditions represented in functional form. Thus, the number of constraints, J, represents the dimension of the terminal curve, surface, or hypersurface. The number of terminal constraints can be any number from one to six for the problem at hand (since there are six independent control parameters).

X_i, $\dot{X}_i$ $(i=1, 2, 3)$: The six state variables for this problem.

λ_i, $\dot{\lambda}_i$ $(i=1, 2, 3)$: The six control variables for this problem (these can be reduced to five independent parameters but a cutoff parameter is added to give six independent control parameters).

P_j $(j=1, \ldots, J)$: Undetermined auxiliary constants that are eliminated in solving for the transversality conditions.

The number of terminal constraints, J, and the number of control parameters determine the number of transversality conditions to be derived from these auxiliary relationships, λ_i and $\dot{\lambda}_i$. This problem has five independent control parameters and a cutoff criteria to be used as control parameters, hence, there can be no more than six independent relationships enforced at the terminal (cutoff) surface.

To visualize what the transversality conditions accomplish, consider the problem where only one terminal constraint is to be satisfied. Since the constraints (for this problem) are independent of the control variables, there is (theoretically) a six parameter family of solutions. It is an enormous task to locate the time optimal solution from this family by surveying trajectories. But the transversality conditions furnish auxiliary boundary conditions that make this task reasonable, although it still requires a lot of computer time. Thus, for one terminal constraint, five associated transversality conditions are derived. When the constraint and the five associated transversality conditions are satisfied simultaneously at cutoff, all the parameters of the problem will have been utilized, and one is assured that the corresponding trajectory is the desired time optimal one.

To illustrate how transversality conditions can be derived, consider J = 1. Thus, F_1 is the only terminal constraint, and it follows that

$$\lambda_1 = -P_1 \frac{\partial F_1}{\partial \dot{x}} \qquad \text{and} \qquad \dot{\lambda}_1 = P_1 \frac{\partial F_1}{\partial x}$$

$$\lambda_2 = -P_1 \frac{\partial F_1}{\partial \dot{y}} \qquad\qquad \dot{\lambda}_2 = P_1 \frac{\partial F_1}{\partial y}$$

$$\lambda_3 = -P_1 \frac{\partial F_1}{\partial \dot{z}} \qquad\qquad \dot{\lambda}_3 = P_1 \frac{\partial F_1}{\partial z}$$

Any one of the six relationships can be solved for P_1 to eliminate that auxiliary quantity, and the results substituted into the other five expressions to yield the five desired transversality conditions. For example,

$$P_1 = -\lambda_1 \Big/ \frac{\partial F_1}{\partial \dot{x}} \,, \quad \lambda_2 = \lambda_1 \left(\frac{\partial F_1}{\partial \dot{y}}\right) \Big/ \left(\frac{\partial F_1}{\partial \dot{x}}\right) \quad \text{or} \quad \lambda_2 \left(\frac{\partial F_1}{\partial \dot{x}}\right) = \lambda_1 \left(\frac{\partial F_1}{\partial \dot{y}}\right) \,, \text{ etc.}$$

so that the five transversality conditions can be written as

$$G_1 = \lambda_2 \frac{\partial F_1}{\partial \dot{x}} - \lambda_1 \frac{\partial F_1}{\partial \dot{y}} = 0$$

$$G_2 = \lambda_3 \frac{\partial F_1}{\partial \dot{x}} - \lambda_1 \frac{\partial F_1}{\partial \dot{z}} = 0$$

$$G_3 = \dot{\lambda}_1 \frac{\partial F_1}{\partial \dot{x}} + \lambda_1 \frac{\partial F_1}{\partial x} = 0$$

$$G_4 = \dot{\lambda}_2 \frac{\partial F_1}{\partial \dot{x}} + \lambda_1 \frac{\partial F_1}{\partial y} = 0$$

$$G_5 = \dot{\lambda}_3 \frac{\partial F_1}{\partial \dot{x}} + \lambda_1 \frac{\partial F_1}{\partial z} = 0$$

At this point, then, the problem is reduced to that of isolating a cutoff time, and initial values for the five independent control parameters such that the constraint (F_1) and the five associated transversality conditions ($G_1, \ldots G_5$) are satisfied at cutoff.

As more constraints are added, the algebra becomes more burdensome, but the procedure is basically unchanged. Thus, for two constraints, F_1 and F_2, two of the six relationships will be used to eliminate P_1 and P_2, resulting in the four required transversality conditions. As more constraints are added, the basic procedures remain the same.

One should note that constraints can be imposed such that there is no way possible for the system to generate a solution trajectory.

B. GENERALIZED TRANSVERSALITIES FOR A PARTICULAR CLASS OF CONSTRAINTS

The relationships

$$\lambda_i = - \sum_{j=1}^{J} P_j \frac{\partial F_1}{\partial \dot{x}_i} \quad \text{and} \quad \dot{\lambda}_i = \sum_{j=1}^{J} P_j \frac{\partial F_j}{\partial x_i}$$

represent, in a sense, the most general expression for transversality conditions associated with the problem at hand. The object now is to specialize this expression somewhat to arrive at an expression for a set of transversality conditions which circumvents most of the work described above for a particularly useful, frequently occurring, class of constraints.

This particular class of constraints is characterized as follows:

i) $J = 3$, i.e. there are three functional terminal constraints, F_1, F_2, and F_3. This results in three auxiliary constants P_1, P_2, and P_3.

ii) The partials of the constraints with respect to the state variables, X_i and $\dot{X}_i$ ($i=1, 2, 3$), must be representable in vector form. In the subsequent development the following notation is used:

$$\frac{\partial F_1}{\partial \dot{x}_i} = \overline{\alpha}, \quad \frac{\partial F_2}{\partial \dot{x}_i} = \overline{\beta}, \quad \frac{\partial F_3}{\partial \dot{x}_i} = \overline{\gamma}$$

and

$$\frac{\partial F_1}{\partial x_i} = \bar{a}, \quad \frac{\partial F_2}{\partial x_i} = \bar{b}, \quad \frac{\partial F_3}{\partial x_i} = \bar{c}.$$

Using some new notation the λ_i and $\dot{\lambda}_i$ relationships can be rewritten using vector and matrix notation as

$$-\bar{\lambda} = \left(\frac{\partial}{\partial \dot{x}_i}\right) \bar{P} = -\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial \dot{x}} & \frac{\partial F_2}{\partial \dot{x}} & \frac{\partial F_3}{\partial \dot{x}} \\ \frac{\partial F_1}{\partial \dot{y}} & \frac{\partial F_2}{\partial \dot{y}} & \frac{\partial F_3}{\partial \dot{y}} \\ \frac{\partial F_1}{\partial \dot{z}} & \frac{\partial F_2}{\partial \dot{z}} & \frac{\partial F_3}{\partial \dot{z}} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

$$\dot{\bar{\lambda}} = \left(\frac{\partial}{\partial x_i}\right) \bar{P} = \begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_2}{\partial x} & \frac{\partial F_3}{\partial x} \\ \frac{\partial F_1}{\partial y} & \frac{\partial F_2}{\partial y} & \frac{\partial F_3}{\partial y} \\ \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial z} & \frac{\partial F_3}{\partial z} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.$$

Now, letting $\left(\frac{\partial}{\partial x_i}\right)^{-1}$ denote the inverse of the matrix $\left(\frac{\partial}{\partial x_i}\right)$, it follows that (assuming the determinant of $\left(\frac{\partial}{\partial x_i}\right)$ does not vanish)

$$\bar{P} = \left(\frac{\partial}{\partial x_i}\right)^{-1} \dot{\bar{\lambda}}$$

and

$$-\bar{\lambda} = \left(\frac{\partial}{\partial \dot{x}_i}\right)\left(\frac{\partial}{\partial x_i}\right)^{-1} \dot{\bar{\lambda}} \quad .$$

Substituting the notations for the partials into this expression and denoting the matrices in a different fashion yields

$$-\bar{\lambda} = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})\ (\bar{a}, \bar{b}, \bar{c})^{-1}\ \dot{\bar{\lambda}} \ .$$

Working through the details to determine $(\bar{a}, \bar{b}, \bar{c})^{-1}$ explicitly reveals that the matrix of cofactors can be written as three vector crossproducts put into row form, namely

$\begin{pmatrix} [\bar{b} \times \bar{c}] \\ [\bar{c} \times \bar{a}] \\ [\bar{a} \times \bar{b}] \end{pmatrix}$, and that the determinant of $(\bar{a}, \bar{b}, \bar{c})$ is given by the scalar quantity

$$\bar{b} \cdot (\bar{c} \times \bar{a}) \ .$$

It follows that

$$-\{\bar{b} \cdot (\bar{c} \times \bar{a})\}\ \bar{\lambda} = (\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \begin{pmatrix} [\bar{b} \times \bar{c}] \\ [\bar{c} \times \bar{a}] \\ [\bar{a} \times \bar{b}] \end{pmatrix} \dot{\bar{\lambda}} \ .$$

Expanding the right side and rearranging terms leads to the following very convenient form for expressing the particular desired transversality conditions:

$$\{\dot{\bar{\lambda}} \cdot (\bar{b} \times \bar{c})\}\ \bar{\alpha} + \{\dot{\bar{\lambda}} \cdot (\bar{c} \times \bar{a})\}\ \bar{\beta} + \{\dot{\bar{\lambda}} \cdot (\bar{a} \times \bar{b})\}\ \bar{\gamma} + \{\bar{b} \cdot (\bar{c} \times \bar{a})\}\ \bar{\lambda} = 0$$

This means that this generalized form (applicable to the particular class of constraints) can be programed and when the user formulates the constraints F_1, F_2, F_3, and the partials denoted by $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{a}$, $\bar{b}$, and $\bar{c}$, he is ready to generate the associated time optimal trajectory (the partials expressions are peculiar to the constraints, so they must be programed with each set of acceptable constraints).

To appreciate how much effort this form for the transversality conditions can save on the part of the engineer, it is sufficient to work through the application (given in the next part of this report) without using the above form. Also, the programming time and computing time is less using the generalized form than it is using the form gotten by performing the indicated operations and reducing to primitive terms.

C. FUNDAMENTAL DISCUSSION OF EARTH-TARGET FUNCTIONAL CUTOFF SURFACES

Before getting into the particular application of the generalized set of transversality conditions, a fundamental discussion is presented on earth-to-target functional constraints. The groundwork leading up to a formulation of "Lunar or Interplanetary Cutoff Surfaces" is presented in Reference 8. The actual formulation of the "Lunar Cutoff Surface" is given in Reference 5.

The essential features of such cutoff surfaces are presented here without delving too deeply into the developmental details.

For any earth-target geometry (any instant in time) a three dimensional volume of geocentric conics exists for travel to that target. The perigee positions of these conics lie on a cone, having its vertex at the earth's center, and each perigee vector of the volume of conics being an element of the cone. The geometry and the desired mission profile which is characterized by vehicle parameters, trajectory shaping, and mission objectives all contribute to the determination of this cone of perigee vector locations and the sizing criteria for the associated conics. The axis of the cone is essentially opposite the vector from earth to target, and is denoted by the unit vector $\hat{M}$. The radius of the cone, or half-cone angle, σ, and the conic sizing criteria, C_3 (twice the total energy per unit mass), are essentially dependent on the parking orbit, the earth-target distance, and the desired flight time to the target. Having made these decisions, in other words having specified these parameters, the elements defining the cone can be treated as constants. Thus, the desired $\hat{M}$, σ, and C_3 can be treated as known quantities, denoted by $\hat{M}^*$, σ^*, and C_3^*.

Now it doesn't really matter that trajectories computed on a realistic trajectory model result in perigee vectors that aren't on a cone for the full three dimensional volume. Feasibility dictates that a relatively small part of the cone is considered for each launch opportunity. Reference 5 shows the applicability of the lunar cutoff surface as described for realistic flight simulations.

D. APPLICATION OF THE PARTICULAR GENERALIZED TRANSVERSALITY CONDITIONS TO THE LUNAR-INTERPLANETARY CUTOFF SURFACE

For a fixed profile there are three constraints to be satisfied at cutoff if a lunar or interplanetary mission is to be fulfilled. These constraints are as follows:

1. $F_1 = V^2 - \frac{2\,GM}{R} - C_3^* = 0$: The injection conic sizing criteria.

2. $F_2 = \hat{M}^* \cdot \bar{S} - e \cos \sigma^* = 0$: Alignment criteria for placing the major axis of the conic as desired.

3. $F_3 = \hat{M}^* \cdot (\overline{R} \times \overline{V}) = 0$: Flight plane alignment criteria. This forces the flight plane at cutoff to contain the vector $\hat{M}^*$.

One should note here that no constraint is present to choose which perigee vector will result, but the constraints do force the perigee vector to be one of those making up the cone. Furthermore, there is no criteria forcing the vehicle to inject at a particular true anomaly, or angular momentum. This is, in fact, the task that the transversality conditions perform, namely, insuring that the vehicle travels the best possible (time optimal) trajectory from the given initial state variables to a set of terminal state variables lying on the functional terminal cutoff surface.

Having formulated the constraints F_1, F_2, and F_3, the next task is the formulation of the partials

$$\frac{\partial F_j}{\partial x_i} \quad \text{and} \quad \frac{\partial F_j}{\partial \dot{x}_i}, \quad \text{as} \quad (j = 1, 2, 3) \quad \text{and} \quad (i = 1, 2, 3).$$

In the constraints listed, $\overline{S}$ denotes a vector in the perigee direction of the cutoff conic, and e the eccentricity of that conic. Minovitch shows in Reference 9 that

$$\overline{S} = \frac{1}{GM} \overline{V} \times (\overline{R} \times \overline{V}) - \hat{R}$$

Rewriting this gives $\overline{S} = \frac{1}{GM} [V^2 \overline{R} - R \dot{R} \overline{V}] - \frac{\overline{R}}{R}$, or

$$\overline{S} = \frac{1}{GM} \left[\left(V^2 - \frac{GM}{R} \right) \overline{R} - R \dot{R} \, \overline{V} \right].$$

From various places [Ref. 15] one can find that

$e = (1 - \frac{p}{a})^{\frac{1}{2}}$ where p is the semilatus rectum, and "a" is the semi-major axis.

This can be rewritten as

$$e = \left[1 + \frac{C_3}{GM} \left(\frac{R^2 V^2 - R^2 \dot{R}^2}{GM} \right) \right]^{\frac{1}{2}}$$

$$e = \left\{ 1 + \frac{1}{GM^2} \quad [C_3 (R^2 V^2 - R^2 \dot{R}^2)] \right\}^{\frac{1}{2}}$$

Now the partials expressions can be developed as follows:

$$F_1 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2\, GM\, (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\frac{\partial F_1}{\partial \dot{x}_i} = 2\, \dot{X}_i \qquad \frac{\partial F_1}{\partial x_i} = 2\, GM\, \frac{X_i}{R^3} \quad ,$$

$$F_2 = \hat{M}^* \cdot \overline{S} - e \cos \sigma^* = M_1^* S_1 + M_2^* S_2 + M_3^* S_3 - e \cos \sigma^*$$

$$\frac{\partial F_2}{\partial \dot{x}_i} = M_1^* \frac{\partial S_1}{\partial \dot{x}_i} + M_2^* \frac{\partial S_2}{\partial \dot{x}_i} + M_3^* \frac{\partial S_3}{\partial \dot{x}_i} - \cos \sigma^* \frac{\partial e}{\partial \dot{x}_i}$$

$$\frac{\partial F_2}{\partial x_i} = M_1^* \frac{\partial S_1}{\partial x_i} + M_2^* \frac{\partial S_2}{\partial x_i} + M_3^* \frac{\partial S_3}{\partial x_i} - \cos \sigma^* \frac{\partial e}{\partial x_i}$$

The intermediate partials expressions can be developed as follows:

$$S_i = \left\{ \frac{1}{GM} \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - GM\, (x^2 + y^2 + z^2)^{-\frac{1}{2}} \right] X_i - \frac{1}{GM} \left[(x\dot{x} + y\dot{y} + z\dot{z})\, \dot{X}_i \right] \right\}$$

$$\frac{\partial S_1}{\partial \dot{x}_i} = \frac{1}{GM} \{ 2\, x\, \dot{X}_i - \dot{x}\, X_i - R\, \dot{R}\, \delta_{1i} \} \quad \text{where } \delta_{1i} = 1,\ 0,\ 0, \text{ as } i = 1,\ 2,\ 3.$$

$$\frac{\partial S_2}{\partial \dot{x}_i} = \frac{1}{GM} \{ 2\, y\, \dot{X}_i - \dot{y}\, X_i - R\, \dot{R}\, \delta_{2i} \} \quad \text{where } \delta_{2i} = 0,\ 1,\ 0, \text{ as } i = 1,\ 2,\ 3.$$

$$\frac{\partial S_3}{\partial \dot{x}_i} = \frac{1}{GM} \{ 2\, z\, \dot{X}_i - \dot{z}\, X_i - R\, \dot{R}\, \delta_{3i} \} \quad \text{where } \delta_{3i} = 0,\ 0,\ 1, \text{ as } i = 1,\ 2,\ 3.$$

$$\frac{\partial S_1}{\partial x_i} = \frac{1}{GM} \left\{ \frac{GM}{R^3}\, x\, X_i - \dot{x}\, \dot{X}_i + \left(V^2 - \frac{GM}{R} \right) \delta_{1i} \right\}$$

$$\frac{\partial S_2}{\partial x_i} = \frac{1}{GM}\left\{\frac{GM}{R^3}\, y\, X_i - \dot{y}\, \dot{X}_i + \left(V^2 - \frac{GM}{R}\right)\delta_{2i}\right\}$$

$$\frac{\partial S_3}{\partial x_i} = \frac{1}{GM}\left\{\frac{GM}{R^3}\, z\, X_i - \dot{z}\, \dot{X}_i + \left(V^2 - \frac{GM}{R}\right)\delta_{3i}\right\}$$

$$e = \left[1 + \frac{1}{GM^2}\left\{C_3\, [(x^2 + y^2 + z^2)\ (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - (x\,\dot{x} + y\,\dot{y} + z\,\dot{z})^2]\right\}\right]^{\frac{1}{2}}$$

$$\frac{\partial e}{\partial \dot{x}_i} = \frac{1}{2\,e}\left\{\frac{1}{GM^2}\left[\frac{\partial C_3}{\partial \dot{x}_i}\ (R^2\, V^2 - R^2\, \dot{R}^2) + C_3\, (2\, R^2\, \dot{X}_i - 2\ R\, \dot{R}\, X_i)\right]\right\}$$

$$= \frac{1}{e}\left\{\frac{1}{GM^2}\left[\dot{X}_i\, (R^2\, V^2 - R^2\, \dot{R}^2 + R^2\, C_3) - C_3\, R\, \dot{R}\, X_i\right]\right\}$$

$$\frac{\partial e}{\partial \dot{x}_i} = \frac{1}{e\, GM^2}\ [(R^2\, V^2 - R^2\, \dot{R}^2 + R^2\, C_3)\ \dot{X}_i - C_3\, R\, \dot{R}\, X_i]$$

$$\frac{\partial e}{\partial x_i} = \frac{1}{2\,e}\left\{\frac{1}{GM^2}\left[\frac{\partial C_3}{\partial x_i}\ (R^2\, V^2 - R^2\, \dot{R}^2) + C_3\, (2\, V^2\, X_i - 2\, R\, \dot{R}\, \dot{X}_i)\right]\right\}$$

$$\frac{\partial e}{\partial x_i} = \frac{1}{e\, GM^2}\left[\frac{GM}{R^3}\ (R^2\, V^2 - R^2\, \dot{R}^2)\ X_i + C_3\, (V^2\, X_i - R\, \dot{R}\, \dot{X}_i)\right]$$

$$\frac{\partial e}{\partial x_i} = \frac{1}{e\, GM^2}\left\{\left[\frac{GM}{R^3}\ (R^2\, V^2 - R^2\, \dot{R}^2) + C_3\, V^2\right] X_i - C_3\, R\, \dot{R}\, \dot{X}_i\right\}$$

Substituting these into the appropriate place yields

$$\frac{\partial F_2}{\partial \dot{x}_i} = \frac{1}{GM} [M_1^* (2 x \dot{X}_i - \dot{x} X_i - R \dot{R} \delta_{1i}) + M_2^* (2 y \dot{X}_i - \dot{y} X_i - R \dot{R} \delta_{2i})$$

$$+ M_3^* (2 z \dot{X}_i - \dot{z} X_i - R \dot{R} \delta_{3i})] - \frac{\cos \sigma^*}{e\, GM^2} [(R^2 V^2 - R^2 \dot{R}^2 + R^2 C_3) \dot{X}_i - C_3 R \dot{R} X_i]$$

$$\frac{\partial F_2}{\partial x_i} = \frac{1}{GM} \left\{ M_1^* \left[\frac{GM}{R^3} x X_i - \dot{x} \dot{X}_i + (V^2 - \frac{GM}{R}) \delta_{1i}\right] + M_2^* \left[\frac{GM}{R^3} y X_i - \dot{y} \dot{X}_i + \right.\right.$$

$$\left.\left. + (V^2 - \frac{GM}{R}) \delta_{2i}\right] + M_3^* \left[\frac{GM}{R^3} z X_i - \dot{z} \dot{X}_i + (V^2 - \frac{GM}{R}) \delta_{3i}\right]\right\}$$

$$- \frac{\cos \sigma^*}{e\, GM^2} \left\{ \left[\frac{GM}{R^3} (R^2 V^2 - R^2 \dot{R}^2) + C_3 V^2\right] X_i - C_3 R \dot{R} \dot{X}_i \right\}.$$

$$F_3 = M_1^* (y \dot{z} - z \dot{y}) + M_2^* (z \dot{x} - x \dot{z}) + M_3^* (x \dot{y} - y \dot{x}) = \hat{M}^* \cdot (\overline{R} \times \overline{V})$$

$$\frac{\partial F_3}{\partial \dot{x}_i} = (\hat{M}^* \times \overline{R})_i$$

$$\frac{\partial F_3}{\partial x_i} = (\hat{V} \times \overline{M^*})_i$$

Now all the partials expressions are established and by eliminating the "i" subscript in favor of vector notation (thus, $X_i = \overline{R}$ as $i = 1, 2, 3$) and rearranging terms, these partials can be written in the required vector form. It follows then that

$$\overline{\alpha} = \frac{\partial F_1}{\partial \dot{x}_i} = 2 \overline{V}$$

$$\bar{\beta} = \frac{\partial F_2}{\partial \dot{x}_i} = \frac{1}{GM} \left\{ 2 (\hat{M}^* \cdot \bar{R}) \bar{V} - (\hat{M}^* \cdot \bar{V}) \bar{R} - R \dot{R} \hat{M}^* \right.$$

$$\left. - \frac{\cos \sigma^* R^2}{e\, GM} [(V^2 + C_3 - \dot{R}^2) \bar{V} - C_3 \dot{R} R \hat{R}] \right\}$$

$$\bar{\gamma} = \frac{\partial F_3}{\partial \dot{x}_i} = (\hat{M}^* \times \bar{R})$$

$$\bar{a} = \frac{\partial F_1}{\partial x_i} = \frac{2\, GM}{R^2} \hat{R}$$

$$\bar{b} = \frac{\partial F_2}{\partial x_i} = \frac{1}{GM} \left\{ \frac{GM}{R} (\hat{M}^* \cdot \hat{R}) \hat{R} - V^2 (\hat{M}^* \cdot \hat{V}) \hat{V} + \left(V^2 - \frac{GM}{R}\right) \hat{M}^* \right.$$

$$\left. - \frac{\cos \sigma^*}{e} \left[\left(V^2 - \dot{R}^2 + \frac{C_3 V^2 R}{GM} \right) \hat{R} - \frac{C_3 R \dot{R} V}{GM} \hat{V} \right] \right\}$$

$$\bar{c} = \frac{\partial F_3}{\partial x_i} = (\bar{V} \times \hat{M}^*)$$

All the information is now available to evaluate the transversality conditions that were developed previously. These were given in the form

$$\{\dot{\bar{\lambda}} \cdot (\bar{b} \times \bar{c})\} \bar{\alpha} + \{\dot{\bar{\lambda}} \cdot (\bar{c} \times \bar{a})\} \bar{\beta} + \{\dot{\bar{\lambda}} \cdot (\bar{a} \times \bar{b})\} \bar{\gamma} + \{\bar{b} \cdot (\bar{c} \times \bar{a})\} \bar{\lambda} = 0$$

The operations indicated here have been performed and reduced to the most primitive terms, but it really doesn't seem useful to include them since it is more economical to program and compute the above form than the primitive form. Furthermore, the work involved in getting to the primitive form is a burdensome exercise.

These results are now in use at MSFC. A program generates the surface criteria, i.e. it determines $\hat{M}^*$, $\cos \sigma^*$, and C_3^* from trajectories that satisfy mission objectives at the target. It then uses these quantities in the cutoff surface constraints and transversality equations to generate optimum trajectories from perturbed initial conditions or vehicle characteristics.

SECTION VI. EFFICIENT TARGETING TECHNIQUES FOR COMPUTING TRAJECTORIES THAT SATISFY MISSION OBJECTIVES

This report, up to this point, has dealt with techniques for terminal constraints in well defined, explicit form. In this part the primary objective is to present efficient techniques for treating terminal constraints that are no more explicit than a mission objective. For example, it is shown how to efficiently represent the periselenum arrival constraints for a trajectory from geocentric parking orbit that passes over a specified lunar location.

The techniques used are made possible by transforming mission objective statements into efficient isolation parameters [Ref. 10], and allowing these desired "efficient parameters" to become slowly varying or floating end conditions. In the following development this is referred to as a "floating end-point" concept. These techniques are employed to generate trajectories from which functional cutoff criteria, such as the cutoff surface quantities $\hat{M}^*$, cos σ^*, and C_3^* (that were used previously in this report), are derived.

A. FUNDAMENTAL DISCUSSION OF THE TRAJECTORY TARGETING OR ISOLATION PROBLEM

The current procedure for isolating or targeting trajectories that satisfy desired terminal conditions is called a multivariable isolation technique or routine. To illustrate what the routine is capable of performing and the basic assumptions involved in its development, the presentation usually proceeds as follows:

Let

$\overline{Q}$ represent the set of Q_k control parameters

$\overline{f}$ represent a set of f_k end conditions.

The trajectory isolation problem, then, is the determination of $\overline{Q}$ that yields the desired values for $\overline{f}$. A Taylor series for some f_i element of $\overline{f}$ is of the form

$$f_i = f_i(\overline{Q}_o) + \left(\frac{\partial f_i}{\partial Q_1}\right)_o (Q_1 - Q_{1_o}) + \ldots + \left(\frac{\partial f_i}{\partial Q_k}\right)_o (Q_k - Q_{k_o}) +$$

$$+ \frac{1}{2}\left[\left(\frac{\partial^2 f_i}{\partial Q_1^2}\right)_o (Q_1 - Q_{1_o})^2 + \ldots + \left(\frac{\partial^2 f_i}{\partial Q_k^2}\right)_o (Q_k - Q_{k_o})^2 + \ldots\right] + \ldots$$

Now using matrix and vector notations, this may be rewritten for all elements of $\bar{f}$ as

$$\bar{f} = \bar{f}\,(\bar{Q}_o) + (\partial_{ij})_o\,\Delta\bar{Q} + \frac{1}{2}\,(\partial^2_{ij})_o\,\Delta\bar{Q}^2 + \ldots,$$

where

$$(\partial_{ij})_o = \begin{bmatrix} \dfrac{\partial f_1}{\partial Q_1} & \dfrac{\partial f_1}{\partial Q_2} \cdots\cdots & \dfrac{\partial f_1}{\partial Q_k} \\ \dfrac{\partial f_2}{\partial Q_1} & \cdots\cdots\cdots\cdots & \\ \vdots & & \\ \dfrac{\partial f_k}{\partial Q_1} & \cdots\cdots\cdots\cdots & \dfrac{\partial f_k}{\partial Q_k} \end{bmatrix}_o, \quad (\partial^2_{ij})_o = \left(\frac{\partial^2 f_i}{\partial Q_j^{\,2}}\right)_o \quad \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, k \end{matrix},$$

etc. At this point a critical assumption is made: "The series may be truncated after the first partials terms." Hence,

$$\bar{f} = \bar{f}\,(\bar{Q}_o) + (\partial_{ij})_o\,(\bar{Q} - \bar{Q}_o)$$

results. This is the basic equation associated with the *so-called multivariable isolation* scheme. Defining the following terms, let $\bar{Q}_o$ represent the set of "best-guess" values for the control parameters, $\bar{f}\,(\bar{Q}_o) = \bar{f}_o$ the associated end conditions, $\bar{f}^*$ the desired end conditions, and $\bar{Q}^*$ the control parameters that result in $\bar{f}^*$, then

$$\bar{f}^* = \bar{f}_o + (\partial_{ij})_o\,(\bar{Q}^* - \bar{Q}_o)$$

results. Let $(\bar{Q}^* - \bar{Q}_o)$, the change needed to go from the $\bar{Q}_o$ to $\bar{Q}^*$, be $\Delta\bar{Q}$, then

$$\Delta\bar{Q} = (\partial_{ij})_o^{-1}\,(\bar{f}^* - \bar{f}_o)$$

results. $(\bar{f}^* - \bar{f}_o)$ is known after the best-guess trajectory is computed. The matrix $(\partial_{ij})_o$ is determined numerically by generating trajectories having discrete, independent variations made in the elements of $\overline{Q}_o$. In other words,

$$\overline{f}\,[Q_{1_o}, \ldots, (Q_{j_o} + \delta Q_{o_j}), \ldots, Q_{k_o}]$$

is established by computing a trajectory. This yields

$$\frac{\partial f_i}{\partial Q_{o_j}}, \; i = 1, \ldots, k, \text{ or } \frac{\partial \overline{f}}{\partial Q_{o_j}} = \frac{\overline{f}\,(Q_{1_o}, \ldots, Q_{j_o} + \delta Q_{o_j}, \ldots, Q_{k_o}) - \overline{f}_o}{\delta Q_{o_j}}$$

One sees, then, that as j goes from $j = 1, \ldots, k$, the complete $(\partial_{ij})_o$ matrix can be numerically approximated, and the equation

$$\Delta\overline{Q} = (\partial_{ij})_o^{-1} \; (\overline{f}^* - \overline{f}_o)$$

can be evaluated to yield $\Delta\overline{Q}$. Now, basically, the trouble with this procedure lies in the truncation of the series past the linear terms. This means that (1) The "best-guess" must be very close to the desired $\overline{Q}^*$ such that the higher order terms (truncated terms) are insignificant, or (2) The parameters being used must in reality be near linear such that the higher order terms are insignificant. Generally speaking neither (1) nor (2) is true, and a great amount of work and computer time is wasted until finally (1) is fulfilled and a solution is found for $\overline{Q}^*$ to give $\overline{f}^*$.

B. PARAMETERS WITH LINEAR FEATURES

In Reference 10, Mr. W. Kizner of JPL presents two parameters, $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$, that are very efficient for isolating desired arrival conditions on lunar or interplanetary trajectories in terms of various departure parameters. References 10 and 12 discuss these parameters in some detail; it is to be pointed out here that these parameters are more linear relative to variations in departure conditions than any others that the author knows about. As was pointed out previously, this linearity feature is of tremendous importance in performing the trajectory isolation problem economically.

The advantages gained by the use of these parameters ($\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$) in solving the trajectory isolation problem are the basis for the transformations to follow. Here some frequent mission constraints are transformed into $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$. By doing this, one can talk about missions in the accepted language, and even feed these into the computer program, but have these transformed into $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$ to keep the advantages that these parameters afford.

C. CONCEPT OF FLOATING END-CONDITIONS

In order to illustrate a basic difficulty involved in the use of $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$, and how the difficulty can be eliminated, a typical example is treated in detail. Suppose one wishes to survey various types of trajectories constrained as follows: All must arrive at a "specified periselenum radius" - R^*_{CA}, at a "specified inclination to the $\overline{ST}$ plane" - I^*_{ST}, and have a "specified flight time from injection to periselenum" - T^*_F. It is known that

$$|\overline{B}| = b = \sqrt{R_{CA} (2 |a| + R_{CA})}, \text{ and } \overline{B} \cdot \hat{T} = b \cos I_{ST}$$

$$\overline{B} \cdot \hat{R} = -b \sin I_{ST}.$$

This implies then that to specify $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$ exactly one must know $|a|$, absolute value of the semi-major axis of the arrival conic, exactly. It is true that $|a|$ varies slowly for small variations in initial conditions but it varies more than enough to cause trouble. In practice, to get $|R_{CA} - R^*_{CA}| < \sim 1.0$ (km) one must get on and off the computer two or three times, each time specifying the $\overline{B}^* \cdot \hat{T}$ and $\overline{B}^* \cdot \hat{R}$ more accurately as the estimate gets better for $|a^*|$. Getting around this difficulty gave rise to the concept of "floating end-point isolation." This concept is useful in isolating other difficult combinations of end-conditions in terms of $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$.

Since one cannot specify $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$ that result in the desired R^*_{CA} and I^*_{ST} to the desired accuracy, consider the possibility of having the program compute these quantities after it generates the best-guess trajectory and establishes the best-guess $|a|$. In other words, the program has a much better estimate of the $|a|$ needed than the user can get without considerable previous work having been done on similar trajectories. Using $|a|$ from the initial trajectory, the specified R^*_{CA}, and I^*_{ST}, the program can compute

$$b^* = \sqrt{R^*_{CA} (2 |a| + R^*_{CA})}, \text{ and } \overline{B}^* \cdot \hat{T} = b^* \cos I^*_{ST}$$

$$\overline{B}^* \cdot \hat{R} = -b^* \sin I^*_{ST}.$$

Any time thereafter, as the isolation proceeds, when the program gets a new "better-guess" trajectory, it recomputes $\overline{B}^* \cdot \hat{T}$ and $\overline{B}^* \cdot \hat{R}$. Now, what does this mean in the fundamental operation of establishing

$$\Delta\overline{Q} = (\partial_{ij})^{-1} (\overline{f}^* - \overline{f}_o) ?$$

This actually allows $\overline{f}^*$ to become a slowly varying function, rather than fixing it as best one could, which was not good enough. Letting $\overline{F}^*$ represent the exact, fixed value that is unknown but desired, then $\overline{f}^* = \overline{F}^* + \overline{\delta f}$, where $\overline{\delta f}$ changes in value, and

$$\Delta\overline{Q} = (\partial_{ij})^{-1} (\overline{F}^* + \overline{\delta f} - \overline{f}_o)$$

results. This could actually converge more rapidly than the fixed end-condition isolation, i.e., $(\overline{F}^* - \overline{f}_o)$ could be greater than $[\overline{F}^* + (\overline{\delta f} - \overline{f}_o)]$ and the difference be harder to eliminate (possibly). The idea is that since the program must find $\Delta\overline{Q}$ anyway, then most likely it is as easy to find the exact desired $\Delta\overline{Q}\ (\overline{F}^* + \overline{\delta f} - \overline{f}_o)$ as it would be to find $\Delta\overline{Q}\ (\overline{f}^* - \overline{f}_o)$. And, since one must go on the computer again and again to refine $\Delta\overline{Q}\ (\overline{f}^* - \overline{f}_o)$ to finally approach $\Delta\overline{Q}\ (\overline{F}^* - \overline{f}_o)$, the floating end-condition isolation looks very favorable.

Here is what is involved in solving the problem using fixed end-condition isolations. Notice that $|a^*|$ is required to specify $\overline{B}^* \cdot \hat{T}$ and $\overline{B}^* \cdot \hat{R}$, and (generally speaking) one does not know $|a^*|$ with sufficient accuracy before running very similar trajectories. So, one makes the best guess that he can for $|a^*|$ (call it a_1), computes $(\overline{B} \cdot \hat{T})_1$, and $(\overline{B} \cdot \hat{R})_1$, and goes on the computer. When the isolation is completed, experience shows that R_{CA} will differ from R^*_{CA} by 10, 20, or perhaps 100 (km) due to the guess for $|a^*|$. One must go on the machine at least once more with a new estimate of $|a^*|$ to refine the results so that $|R_{CA} - R^*_{CA}|$ takes on a sufficiently small value to suit one's purposes.

How does the floating end-condition isolation help in this problem? The idea is to let the program use the $|a|$ that it gets on the "best-guess" trajectory rather than trying to input one. Then as the isolation progresses and "better" trajectories are generated, discard the old $|a|$ and update $(\overline{B} \cdot \hat{T})$ and $(\overline{B} \cdot \hat{R})$ using $|a|$ from these "better" trajectories. Thus, when the floating end-condition is finally isolated, $|R_{CA} - R^*_{CA}|$ is within whatever tolerance the user has specified - no more refinements are needed. The user has not been bothered trying to estimate what $|a^*|$ should have been as when using fixed end-condition isolations.

In actual practice, experience has shown that the floating end-condition for lunar flight studies takes no more trajectories per case to isolate than a fixed end-condition isolation. It has made the work much easier, faster, and saved a considerable amount of manhours and computer time for the type problem just discussed.

D. TRANSFORMING CONSTRAINTS INTO THE LINEAR PARAMETERS

Using the parameters $\overline{B} \cdot \hat{T}$ and $\overline{B} \cdot \hat{R}$, and the concept of floating end-conditions, various constraints are transformed into an efficient system for computations. These transformations are presented in detail.

1. Arrival at Periselenum with $R_{CA} = R^*_{CA}$, $I_{ST} = I^*_{ST}$, $T_F = T^*_F$

This particular set of constraints was discussed in the previous example. As is shown in References 10 and 11,

$$b^* = \sqrt{R^*_{CA}\,(2\,|a| + R^*_{CA})}\,, \qquad \text{and} \qquad \begin{aligned} \overline{B}^* \cdot \hat{T} &= b^* \cos I^*_{ST} \\ \overline{B}^* \cdot \hat{R} &= -b^* \sin I^*_{ST}\,. \end{aligned}$$

Thus the constraints are represented in the most nearly linear parameters that are available, except for flight time, T_F. Flight time is still a nonlinear function of variations in initial conditions. JPL has done some research [12] on a linearized flight time parameter, but no implementation has been done here in this area.

2. Arrival on Orbits Inclined at a Minimum to Some Reference Plane

The latitude or declination of $\hat{S}$ relative to some plane is the value of the minimum orbital inclination that can be established relative to that same plane. Consider the lunar equator as the basic reference plane to which minimum orbital inclination is desired. The principles carry over identically to any other reference plane. Transformations are defined [Ref. 13] and are available for establishing $\hat{S}$ in selenographic coordinates. Define $\overline{T}$ in the lunar equator, normal to $\hat{S}$, as $\overline{T} = (S_y, -S_x, 0)$. This then means that arrival in the selenographic referenced $\hat{S}\,\hat{T}$ plane is, in fact, arrival in the orbit plane inclined at a minimum to the lunar-equator. Thus,

$$\text{Specify:} \begin{cases} R^*_{CA},\ I^*_{ST} = 0° & \text{(arrives in the } \hat{S}\hat{T} \text{ plane, "direct" motion along orbit)} \\ \\ R^*_{CA},\ I^*_{ST} = 180° & \text{(arrives in the } \hat{S}\hat{T} \text{ plane, "rectrograde" motion along orbit)} \end{cases}$$

E. CIRCUMLUNAR FREE RETURN PROFILES

This implies that the outbound transit is designed such that the vehicle can travel from injection, pass near the moon on the side away from the earth, and continue to satisfactory earth reentry with no postinjection thrust applications. Other reasonable constraints are often superimposed upon this basic profile.

The sensitivity of earth reentry conditions to launch or injection conditions is well known for circumlunar trajectories. The concept here is to isolate conditions at periselenum that are known to be near free return conditions. The procedures for refining these to result in acceptable reentry conditions is not given since these refinements represent very small changes in the departure parameters.

Now, how are free return periselenum conditions formulated? Miele [Ref. 14] shows that (in the restricted three-body problem) any trajectory arriving with periselenum on the earth-moon line is symmetric with respect to the earth-moon line. Also, any trajectory having periselenum in the xz plane of the classical restricted three-body problem (plane containing the earth, moon, and the pole of their motion about the barycenter, at any time) with a 90° or 270° azimuth is symmetric with respect to the xz plane. In other words, if periselenum is on the earth-moon line, the velocity may have any local azimuth direction and still retain the outbound-to-inbound symmetry about the earth-moon line. However, if periselenum moves away from the earth-moon line but stays in the xz plane of the restricted three-body problem (analogous to moving in latitude referenced to the travel plane of the moon), then the velocity must have a local azimuth direction (relative to the pole of the moon's motion about the earth) of 90° or 270° to keep the outbound-to-inbound symmetry with respect to the xz plane. Better criteria exists for approximating free return arrival periselenum conditions but this will suffice to illustrate how the transformation can be used.

Notice that free returns having periselenum on the earth-moon line represent a subset of the more general problem in which the periselenum location is specified arbitrarily. The other type of free return is likewise a subset of the more general problem in which periselenum location and the azimuth direction are specified arbitrarily. Transformations are developed for the more general problems of which the free returns are particular situations.

F. TRANSFORMING CONSTRAINTS THAT HAVE CIRCUMLUNAR FREE RETURNS AS PARTICULAR SITUATIONS

In all of these transformations assume that the vehicle must be constrained to circumnavigate the moon and pass at a specified close approach distance. In the notation to be used here, this means that R_{CA} must be forced to a specified desired value, R^*_{CA}.

1. Arrival at a Desired Periselenum Position (R^*_{CA}, ϕ^*, λ^*)

Free returns having periselenum on the earth-moon line are a particular family out of this class of trajectories.

The JPL space program [Ref. 15] has the direction of the moon from the earth available in various coordinate systems; call this direction $\hat{E}$ (ϕ_E, λ_E). The conditions to be imposed then are denoted by

$$b^* = \sqrt{R^*_{CA}\,(2\,|a| + R^*_{CA})}$$

$$\hat{P}^* = \hat{E}\,.$$

Using a common coordinate system throughout, and treating $\hat{S}$ as a slowly varying function, the unit normal to the desired flight plane is given by

$\overline{N} = \hat{S} \times \hat{E}$, where $\hat{S}$ denotes the direction of the incoming asymptote,

and

$$\hat{N}^* = \overline{N}/(N_x^2 + N_y^2 + N_z^2)^{\frac{1}{2}}$$

The desired direction of the impact parameter [Ref. 10] is given by

$$\hat{B}^* = \hat{N}^* \times \hat{S}$$

and finally $\overline{B}^* = b^* \cdot \hat{B}^*$ from which two desired parameters are computed,

$(\overline{B}^* \cdot \hat{T})$ and $(\overline{B}^* \cdot \hat{R})$.

These parameters orient the flight plane as desired but $\hat{P}^*$ is not yet forced to the desired location in that plane. This is accomplished by enforcing

$$1 - (\hat{P}^* \cdot \hat{P}) = 0 \quad \text{or} \quad 1 - (\hat{P} \cdot \hat{E}) = 0.$$

From the fundamental principles [Ref. 11] it is known that these constraints, $(\overline{B}^* \cdot \hat{T})$, $(B^* \cdot \hat{R})$, and $(1 - (\hat{P} \cdot \hat{E}) = 0)$, can be satisfied with variations in

T_L (booster lift-off time)

ΔT_c (coast time in earth parking orbit)

ΔT_B (time of lunar transit injection burn from the earth parking orbit).

2. Arrival at a Desired Periselenum Position (R^*_{CA}, ϕ^*, λ^*) with a Desired Velocity Direction (Σ^*)

For free returns of this type the periselenum must occur in the xz plane of the restricted three-body problem with a velocity heading normal to that plane.

The coordinate system for this transformation is selenocentric and can be oriented arbitrarily so long as periselenum position and velocity, and the earth's position and velocity relative to the moon are known in the same system. Assume that periselenum position, $\bar{\rho}$, and velocity, $\dot{\bar{\rho}}$, moon's position, $\bar{R}_m$, and velocity, $\dot{\bar{R}}_m$, are known in the same geocentric space-fixed coordinate system, then

$\bar{R}_s = \bar{\rho} - \bar{R}_m$ gives selenocentric periselenum position,

and

$\bar{V}_s = \dot{\bar{\rho}} - \dot{\bar{R}}_m$ gives selenocentric periselenum velocity.

The normal to the plane of the moon's motion about the earth is given by $\bar{N}_m = \bar{R}_m \times \dot{\bar{R}}_m$. Normalize $\bar{R}_m$, and $\bar{N}_m$, establishing $\hat{R}_m$ and $\hat{N}_m$; then $\hat{E}_m = \hat{N}_m \times \hat{R}_m$, and $\hat{R}_m$, $\hat{E}_m$, and $\hat{N}_m$ is an orthogonal, right-handed system of unit vectors. The $\hat{R}_m$ $\hat{N}_m$ plane is analogous to the xz plane of the restricted three-body problem.

The JPL program has an option allowing $\hat{S}$, $\hat{T}$, $\hat{R}$, and $\bar{B}$ to be referenced to the $\hat{R}_m$, $\hat{E}_m$, and $\hat{N}_m$ system. This option should be used for the type problem being considered.

For this problem transform the desired conditions into the desired $\bar{B}$ relative to the $\hat{R}_m$, $\hat{E}_m$, $\hat{N}_m$ system. Define

$$\bar{\rho}_m = \begin{pmatrix} \hat{R}_s \cdot \hat{R}_m \\ \hat{R}_s \cdot \hat{E}_m \\ \hat{R}_s \cdot \hat{N}_m \end{pmatrix} , \quad \dot{\bar{\rho}}_m = \begin{pmatrix} \hat{V}_s \cdot \hat{R}_m \\ \hat{V}_s \cdot \hat{E}_m \\ \hat{V}_s \cdot \hat{N}_m \end{pmatrix} ;$$

thus, $\bar{\rho}_m$ and $\dot{\bar{\rho}}_m$ denote the vehicle's selenocentric position and velocity, respectively, relative to the system of unit vectors $\hat{R}_m$, $\hat{E}_m$, $\hat{N}_m$ at periselenum arrival time. Letting

$$\sin\delta = -\frac{\rho_{mz}}{\sqrt{\rho_{mx}^2 + \rho_{my}^2 + \rho_{mz}^2}} , \qquad \cos\delta = -\frac{\sqrt{\rho_{mx}^2 + \rho_{my}^2}}{\sqrt{\rho_{mx}^2 + \rho_{my}^2 + \rho_{mz}^2}}$$

$$\sin\alpha = \frac{\rho_{my}}{\sqrt{\rho_{mx}^2 + \rho_{my}^2}} , \qquad \cos\alpha = \frac{\rho_{mx}}{\sqrt{\rho_{mx}^2 + \rho_{my}^2}}$$

then

$$\hat{\rho}_m = \begin{pmatrix} \cos\delta\cos\alpha^* \\ \cos\delta\sin\alpha^* \\ \sin\delta \end{pmatrix} , \quad \dot{\hat{\rho}}_m = \begin{pmatrix} -\sin\delta\cos\alpha^*\cos\Sigma^* - \sin\alpha^*\sin\Sigma^* \\ -\sin\delta\sin\alpha^*\cos\Sigma^* + \cos\alpha\sin\Sigma^* \\ \cos\delta\cos\Sigma^* \end{pmatrix}$$

represent periselenum position and velocity in terms of the desired quantities α^* and Σ^*, α^* and Σ^* being specified. Some free returns, as was pointed out, are just a subclass of this general formulation where $\alpha^* = 0$, and $\Sigma^* = 270°$. Notice now that $\hat{\delta}$ is still a free parameter. The objective is to choose δ such that the flight plane contains $\hat{S}_m$.

The first task is to determine δ such that the flight plane, $\hat{\rho}_m\ \dot{\hat{\rho}}_m$, contains $\hat{S}_m$. This means that δ must be chosen such that

$$\hat{N}_m = \dot{\hat{\rho}}_m \times \hat{\rho}_m, \text{ and } \hat{N}_m \cdot \hat{S}_m = 0.$$

These operations yield

$$\hat{N}_m = \begin{bmatrix} -\sin\alpha^* \cos\Sigma^* + \sin\delta \cos\alpha^* \sin\Sigma^* \\ \cos\alpha^* \cos\Sigma^* + \sin\delta \sin\alpha^* \sin\Sigma^* \\ -\cos\delta \sin\Sigma^* \end{bmatrix}$$

and

$$\hat{N}_m \cdot \hat{S}_m = 0 = \sin\delta\,\Sigma^* \,(S_x \cos\alpha^* + S_y \sin\alpha^*) - \cos\delta\, S_z \sin\Sigma^*$$

$$+ \cos\Sigma^* \,(S_y \cos\alpha^* - S_x \sin\alpha^*) .$$

Letting

$$P = S_x \cos\alpha^* + S_y \sin\alpha^* \quad \text{and} \quad M = S_y \cos\alpha^* - S_x \sin\alpha^*$$

and solving the $\hat{N}_m \cdot \hat{S}_m = 0$ equation for $\sin\delta$ one gets

$$\sin\delta^* = \frac{-\cos\Sigma^* \cdot P \cdot M - S_z \,[\sin^2\Sigma^* \,(P^2 + S_z^2) - \cos^2\Sigma^* \, M^2]^{\frac{1}{2}}}{\sin\Sigma^* \,(P^2 + S_z^2)} .$$

This shows that Σ^* of 0° and π give trouble; so, for present purposes these are arbitrarily ruled out. By definition, $-\frac{\pi}{2} \leq \delta \leq \frac{\pi}{2}$ so that

$$\cos\delta^* = (1 - \sin^2\delta^*)^{\frac{1}{2}}$$

Finally, using the desired declination, δ^*, one can compute

$$\hat{B}^*_m = \hat{N}^*_m \times \hat{S}_m$$

The vector $\hat{B}^*_m$ orients the flight plane as desired relative to the moon's travel plane. Now the periselenum position vector, $\hat{\rho}_m$, must be moved to the desired position, $\hat{\rho}^*_m$. This implies that

$$1 - (\hat{\rho}^*_m \cdot \hat{\rho}_m) = 0$$

The conditions $(\overline{B}^* \cdot \hat{T})$, $(\overline{B}^* \cdot \hat{R})$, and $[1 - (\hat{\rho}^*_m \cdot \hat{\rho}_m) = 0]$, can be satisfied with variations in T_L (booster lift-off time), ΔT_c (coast time in the geocentric parking orbit), and ΔT_B (burn time of the lunar transit injection stage), but variations in T_F (flight time from lunar transit injection to periselenum) must be acceptable.

3. Arrival at a Desired Periselenum Position and Passing Over Another Specific Site

As was true in the previous developments for free return profiles, the transformation developed here treats a general problem of which the free return profile is a particular case.

In this problem two positions are specified on the arrival orbit, thus, uniquely specifying the desired orbit projection on the lunar surface. The problem now is to move the $\hat{S}$, incoming asymptote, such that it is brought into the desired orbit plane. This cannot be done, in general, with small variations in initial conditions. Let

$$\hat{P}^* = \begin{bmatrix} \cos\phi^*_P \cos\alpha^*_P \\ \cos\phi^*_P \sin\alpha^*_P \\ \sin\phi^*_P \end{bmatrix}$$

denote the desired periselenum direction, and

$$\hat{R}^*_1 = \begin{bmatrix} \cos\phi^*_1 \cos\alpha^*_1 \\ \cos\phi^*_1 \sin\alpha^*_1 \\ \sin\phi^*_1 \end{bmatrix}$$

denote the other desired position on the orbit, then

$$\bar{N} = \hat{R}_1^* \times \hat{P}^*$$

and

$$\hat{N}^* = \bar{N}/(N_x^2 + N_y^2 + N_z^2)^{\frac{1}{2}}$$

is completely specified. If $\hat{S}$ is to be made to lie in the $\hat{P}^*$ $\hat{R}_1^*$ plane, the relationship

$$\hat{N}^* \cdot \hat{S} = 0$$

must be satisfied. Furthermore, if $\hat{S}$ keeps the unit magnitude then

$$\hat{S} \cdot \hat{S}^* = 1$$

must hold.

Now, what reasonable procedures move $\hat{S}$ such that these two relationships are satisfied?

It is known from experience [Ref. 11] that T_F variations represent one means of bringing $\hat{S}$ into $\hat{S}^*$, i.e., causing $\hat{S}$ to move into the $\hat{P}^*$ $\hat{R}_1^*$ plane, there being denoted as $\hat{S}^*$. This procedure may or may not be expensive. For example if a relatively short flight time is required, then the injection energy requirements go up accordingly.

Using T_F variations to bring about $\hat{S} = \hat{S}^*$ implies that $S_z \approx S_z^*$, hence assume $S_z = S_z^*$. Then

$$S_x^* = \pm (1 - S_z^{*2} - S_y^{*2})^{\frac{1}{2}} \quad \text{and} \quad N_x^* S_x^* + N_y^* S_y^* + N_z^* S_z^* = 0$$

can be used to determine S_x^* and S_y^*. This results in

$$S_y^* = \frac{- N_y^* N_z^* S_z^* \pm N_x^* (N_x^{*2} + N_y^{*2} - S_z^{*2})^{\frac{1}{2}}}{N_x^{*2} + N_y^{*2}}$$

and

$$S^*_x = \pm (1 - S^{*2}_z - S^{*2}_y)^{\frac{1}{2}}$$

Choosing the $\hat{R}_m$, $\hat{E}_m$, $\hat{N}_m$ reference system, analogous to the x, y, z axes (respectively), circumlunar flights result always in $S_x > 0$. These equations, as usual, are not completely general. It is evident that $(N^{*2}_x + N^{*2}_y = 0)$ cannot be permitted. This means that the two specified points on the desired orbit must not both be in the travel plane of the moon.

To complete the development, $\hat{T}^*$ and $\hat{R}^*$ follow from $\hat{S}^*$,

$$\hat{B}^* = \hat{N}^* \times \hat{S}^*, \; b^* = \sqrt{R^*_{CA} \; (2 \; |a| \; + R^*_{CA})}, \text{ and } \bar{B}^* = b^* \; \hat{B}^*,$$

from which $(\bar{B}^* \cdot \hat{T})$ and $(\bar{B}^* \cdot \hat{R})$ can be computed.

Another means for forcing $\hat{S}$ into $\hat{S}^*$ is by making powered plane changes. Needless to say, such maneuvers can be prohibitively expensive. One easy way to formulate this procedure is to take the projection of $\hat{S}$ in the $\hat{P}^*$ $\hat{R}^*_1$ plane as the direction of $\hat{S}^*$. This means that

$$\bar{S}^* = \hat{S} - (\hat{S} \cdot \hat{N}^*) \; \hat{N}^*$$

$$\hat{S}^* = \bar{S}^* / (S^{*2}_x + S^{*2}_y + S^{*2}_z)^{\frac{1}{2}},$$

and

$$\hat{B}^* = \hat{N}^* \times \hat{S}^*$$

The unit vectors $\hat{T}$ and $\hat{R}$ follow from $\hat{S}^*$, and

$$b^* = [R^*_{CA} \; (2 \; |a| \; + R^*_{CA})]^{\frac{1}{2}}$$

from which $(\bar{B}^* \cdot \hat{T})$ and $(\bar{B}^* \cdot \hat{R})$ follow immediately.

The remaining condition

$$1 - (\hat{P} \cdot \hat{P}^*) = 0$$

forces the periselenum vector to the desired position, $\hat{P}^*$.

SECTION VII. CONCLUSIONS AND RECOMMENDATIONS

The development presented in this report is in four general areas, as is pointed out in the abstract. Conclusions and recommendations relative to each general area are listed as follows:

1. Differential Equations of the Control Variables

If earth oblateness terms are included in the acceleration of the vehicle due to gravity, then the primary influence of the oblateness terms on the control variables should be included. Essentially no extra computing time is required due to including the dominant part of the oblateness influence rather than just the spherical earth influence (see pages 16 and 17).

2. Parametric Representation of the Control Variables

Detailed transformations are given relating three parameter sets for representing the control variables. These parameter sets are referred to as the "ALPHAS," (α_n, α_w, $\dot{\alpha}_n$, $\dot{\alpha}_w$, Λ, $\dot{\Lambda}$), the "CHIS," (χ_p, χ_y, $\dot{\chi}_p$, $\dot{\chi}_y$, Λ, $\dot{\Lambda}$), and the "LAMBDAS," ($\bar{\lambda}$, $\dot{\bar{\lambda}}$). These are simply three distinct sets of quantities defining the thrust vector, and its time rate of change. The transformations relating these parameters allow the user to take advantage of the facility either set might afford for performing a particular task.

It is shown that "control variable initial conditions" should be given in terms of the "ALPHAS." However, the isolation of "control variable initial conditions" that result in trajectories to desired terminal conditions (using a multivariable isolation scheme) should be done in terms of the "LAMBDAS." By utilizing the transformations relating the "ALPHAS" and "LAMBDAS" one can specify the initial conditions efficiently as "ALPHAS," transform to "LAMBDAS," and perform the isolation efficiently in terms of the "LAMBDAS."

It is recommended that a survey be conducted using a simplified physical model to determine initial "ALPHAS" required for various vehicles to leave various parking orbits and burn into various desired terminal conic sections. The resulting data should be represented functionally using curve-fitting techniques. These functions can then be used in more refined programs, thereby relieving the engineer of the task of estimating "control variable initial conditions," eliminating some duplication of effort since many engineers do this task independently at many different places, and saving a sizable amount of computer time that is usually wasted due to poor estimates of the "control variable initial conditions."

3. Transversality Conditions and Lunar-Interplanetary Mission Cutoff Surfaces

The fundamental nature of transversality conditions and cutoff surfaces is discussed, and a detailed application of these concepts to lunar-interplanetary flights is given.

The lunar-interplanetary cutoff surface constraints belong to a class of terminal constraints characterized as follows:

i) There are three functional terminal constraints; call them F_1, F_2, and F_3.

ii) The partials of these constraints with respect to the state variables X_i and $\dot{X}_i$ $(i = 1, 2, 3)$ can be represented in vector form. For the present development the notation is introduced such that as $(i = 1, 2, 3)$,

$$\frac{\partial F_1}{\partial \dot{X}_i} = \overline{\alpha}\ , \quad \frac{\partial F_2}{\partial \dot{X}_i} = \overline{\beta}\ , \quad \frac{\partial F_3}{\partial \dot{X}_i} = \gamma$$

$$\frac{\partial F_1}{\partial X_i} = \overline{a}\ , \quad \frac{\partial F_2}{\partial X_i} = \overline{b}\ , \quad \frac{\partial F_3}{\partial X_i} = \overline{c}\ .$$

It is shown that for this class of terminal constraints, the transversality condition may be written as

$$\{\dot{\overline{\lambda}} \cdot (\overline{b} \times \overline{c})\}\ \overline{\alpha} + \{\dot{\overline{\lambda}} \cdot (\overline{c} \cdot \overline{a})\}\ \overline{\beta} + \{\dot{\overline{\lambda}} \cdot (\overline{a} \times \overline{b})\}\ \overline{\gamma} + \{\overline{b} \cdot (\overline{c} \times \overline{a})\}\ \overline{\lambda} = 0\ ,$$

where $\overline{\lambda}$, and $\dot{\overline{\lambda}}$ represent the control variables for this problem. It is more economical to compute the transversality conditions in this form than it is to reduce this expression to so-called primitive terms.

4. Isolating or Targeting Trajectories that Satisfy Mission Objectives

Parameters that are used to define mission objectives are not normally efficient for use in isolation or targeting routines. Hence, procedures are given for transforming mission parameters into efficient isolation parameters. All the parameters treated in this report involve what is called a "floating end-point concept." This simply means that the desired terminal values of the efficient isolation parameters have some variation as initial conditions are varied, but experience has shown that this variation presents no problem. On the other hand, introduction of this "floating end-condition concept" results in significant savings in man-hours and computer time when isolating earth departure conditions that result in trajectories that fulfill lunar or interplanetary mission objectives.

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